Newton's Method

Estimate $\sqrt{23}$.

$\sqrt{23} = x$

$23 = x^2$

$0 = x^2 - 23$

Therefore find the zeros of $f(x) = x^2 - 23 \implies f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \Rightarrow \quad x_{n+1} = x_n - \frac{x_n^2 - 23}{2x_n} = \frac{1}{2}x_n + \frac{23}{2x_n}$$

$x_0 = 5$ (a guess) (off by about 2.04×10^{-1})

$$x_1 = \frac{1}{2} \cdot 5 + \frac{23}{2 \cdot 5} = \frac{24}{5} = 4.8$$ (off by about 4.17×10^{-3})

$$x_2 = \frac{1}{2} \cdot \frac{24}{5} + \frac{23}{2 \cdot \frac{24}{5}} = \frac{1151}{240} = 4.7958\overline{3}$$ (off by about 1.81×10^{-6})

$$x_3 = \frac{1}{2} \cdot \frac{1151}{240} + \frac{23}{2 \cdot \frac{1151}{240}} = \frac{2649601}{552480} \approx 4.79583152331306\ldots$$ (off by about 3.42×10^{-13})

$$x_4 = \frac{1}{2} \cdot \frac{2649601}{552480} + \frac{23}{2 \cdot \frac{2649601}{552480}} = \frac{14040770918401}{2927703120960}$$ (off by about 1.22×10^{-26})

$$x_5 = \frac{1}{2} \cdot \frac{14040770918401}{2927703120960} + \frac{23}{2 \cdot \frac{14040770918401}{2927703120960}} = \frac{394286495966030522000793601}{82214417676974026385569920}$$

(off by about 1.54×10^{-53})