Survey of Calculus

Exercise: Consider the function $f(x) = \sqrt{x+3}$. Find the slope and the equation of the line tangent to f at $x = 13$.

Solution: Recall, the slope of the tangent line is given by,

$$\text{Slope of Tangent } = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

In this case, we can simply plug into $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Remember! $f(x+h)$ is found by replacing each x in f with $x+h$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{(x+h)+3} - \sqrt{x+3}}{h}$$

In order to evaluate this limit, we need to *multiply the numerator and denominator by the conjugate*. This is a good strategy when you have a binomial and one or both of the terms involve roots. The *conjugate* is found by changing the sign of the second term. For example, the conjugate of $a - b$ is $a + b$. This will rationalize our expression.

$$\lim_{h \to 0} \frac{\sqrt{(x+h)+3} - \sqrt{x+3}}{h} = \lim_{h \to 0} \frac{\sqrt{(x+h)+3} - \sqrt{x+3}}{h} \cdot \frac{\sqrt{(x+h)+3} + \sqrt{x+3}}{\sqrt{(x+h)+3} + \sqrt{x+3}}$$

When evaluating $\frac{\sqrt{(x+h)+3} - \sqrt{x+3}}{h} \cdot \frac{\sqrt{(x+h)+3} + \sqrt{x+3}}{\sqrt{(x+h)+3} + \sqrt{x+3}}$, we F.O.I.L. the numerator, and typically want to leave the denominator unsimplified, as we will be able to cancel out our h term later. If the rational expression is in the denominator, we will want to leave the numerator unsimplified.

First, we will evaluate $(\sqrt{(x+h)+3} - \sqrt{x+3})(\sqrt{(x+h)+3} + \sqrt{x+3})$. Then we will evaluate the limit. Recall, $(a-b)(a+b) = a^2 - b^2$.

$$\begin{aligned}(\sqrt{(x+h)+3} - \sqrt{x+3})(\sqrt{(x+h)+3} + \sqrt{x+3}) &= (\sqrt{(x+h)+3})^2 - (\sqrt{x+3})^2 \\ &= ((x+h)+3) - (x+3) \\ &= x + h + 3 - x - 3 \\ &= h.\end{aligned}$$

Plugging back into the limit,

$$\lim_{h\to 0}\frac{\sqrt{(x+h)+3}-\sqrt{x+3}}{h}\cdot\frac{\sqrt{(x+h)+3}+\sqrt{x+3}}{\sqrt{(x+h)+3}+\sqrt{x+3}} = \lim_{h\to 0}\frac{h}{h(\sqrt{(x+h)+3}+\sqrt{x+3})}$$
$$= \lim_{h\to 0}\frac{1}{(\sqrt{(x+h)+3}+\sqrt{x+3}}$$
$$= \frac{1}{\sqrt{(x+(0))+3}+\sqrt{x+3}}$$
$$= \frac{1}{\sqrt{x+3}+\sqrt{x+3}}$$
$$= \frac{1}{2\sqrt{x+3}}.$$

Therefore, $f'(x)=\dfrac{1}{2\sqrt{x+3}}$. This tells us, for example, that at $x=13$, when

$$f(13)=\sqrt{13+3}=\sqrt{16}=4,$$

the slope is changing at a rate of

$$f'(13)=\frac{1}{2\sqrt{13+3}}=\frac{1}{2\sqrt{16}}=\frac{1}{2\cdot 4}=\frac{1}{8}.$$

This is the *derivative* of f at $x=13$, which tells us the slope of the tangent line at $x=13$.

Now, we have our two pieces of information needed to find the equation of the line tangent to f at $x=13$; our point $(13,4)$, and the slope, $\dfrac{1}{8}$. Using the point-slope equation, we have,

$$(y-y_1)=m(x-x_1)$$
$$(y-4)=\frac{1}{8}(x-13)$$
$$y-4=\frac{1}{8}x-\frac{13}{8}$$
$$y=\frac{1}{8}x-\frac{13}{8}+4$$
$$=\frac{1}{8}x+\frac{19}{8}.$$

Thus, the equation of the line tangent to f at $x=13$ is represented by $y=\dfrac{1}{8}x+\dfrac{19}{8}$.