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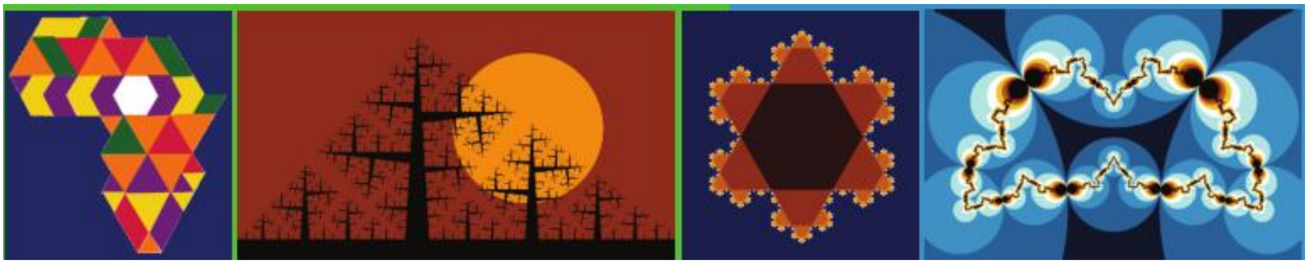


Mathematics and Art Activity - Basic Plane Tessellation with GeoGebra

Worksheet: Exploring Regular Edge-Edge Tessellations of the Cartesian Plane and the Mathematics behind it.

Goal: To enable Maths educators to use GeoGebra to understand some of the mathematics that supports the construction of regular plane tessellations. References to results that are aligned to the CAPS mathematics curriculum will also be discussed .

Relevant Maths Keywords and Concepts:
Tessellation or Tiling, Euclidean Plane, Regular shape, Irregular shape, Polygon, Interior angles, Exterior angles, Convex and Concave quadrilateral, Congruent shapes, Regular Tessellation, Irregular Tessellation, Semi-regular Tessellation.

1. Pre-Knowledge for exploring basic plane tessellations:

Polygons are 2-dimensional shapes. They are made of straight lines (edges), and the shape is "closed" (all the lines connect up in vertices).

Regular Polygons are polygons that are equiangular (all angles are equal in measure) and equilateral (all sides have the same length).

You should also know that:

- a whole turn around any point on a surface is 360°
- the sum of the internal angles of any triangle = 180°
- the sum of the internal angles of any quadrilateral = 360°
- the sum of the external angles of any polygon=360° (one whole turn)
- the sum of the interior angles of a n-sided regular polygon = (n - 2) × 180°
- how to calculate or measure the interior angles of regular polygons

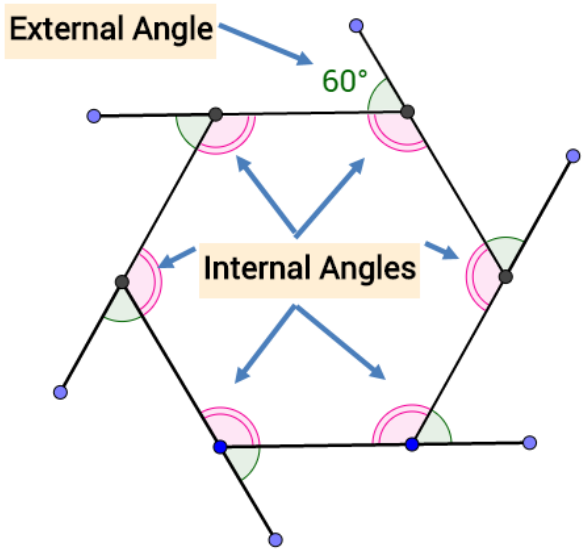
Internal and External Angles of a Regular Polygon

Example:

The sum of the internal angles of a regular Hexagon (n=6) is

(n - 2) × 180°=(6 - 2) × 180°=720°.

Hence the internal angles of a regular Hexagon is $\frac{720°}{6} = 120°$.



Symmetry

This is the property that a figure coincides with itself under an isometry, where an **isometry** is an action (movement) the preserves size and shape.

There are three basic types of isometries that present symmetry of a figure in a plane.

Types of Symmetry:

(a) **Reflectional symmetry.** An object has reflectional symmetry if you can reflect it in a way such that the resulting image coincides with the original. Hold a mirror up to it, its reflection looks identical.

(b) **Rotational symmetry.** An object has rotational symmetry if it can be rotated about a point in such a way that its rotated image coincides with the original figure before turning a full $360°$.

(c) **Translational symmetry.** An object has translational symmetry if you move it along a straight path without turning it to produce the same image.

Defining a Tessellation

A tessellation can be defined as the covering of a plane with a repeating unit consisting of one or more shapes (regular or irregular) in such a way that:

- there are no open spaces between and no overlapping of the shapes that are used;
- the covering process has the potential to continue indefinitely (for a surface of infinite dimensions – Cartesian Plane).

2. Regular Tessellations of the Plane

Tessellations in which one regular polygon is used repeatedly are called **regular tessellations.**

Two key questions to consider –

Which regular polygons will tessellate (or tile) the plane and why?

How many different tessellations are possible in each case?

Some Terminology about Naming of Plane Tessellations:

Consider the example of an edge-edge plane tessellation in Figure 1. Although all the polygons are regular, there are more than one type of polygon which that are used to tessellate. This makes this a non-regular tessellation (or tiling) of the plane.

A **vertex** is a common point where sides (edges) of polygons meet. The **configuration of a vertex** is the sequence of polygon orders that exist around it. Normally these orders are given in a sequence starting with the lowest order.

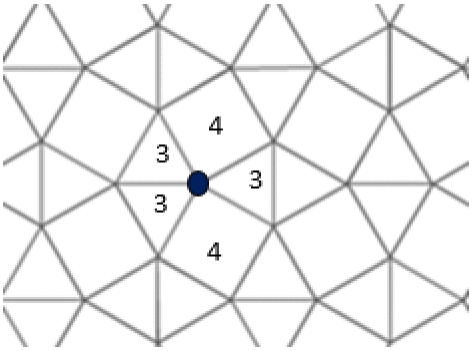


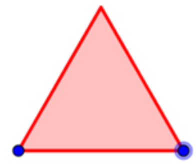
Figure 1. **3.3.4.3.4** Tessellation

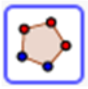
The vertex configuration of each vertex in the tiling shown in Figure 1. is **3.3.4.3.4** as each vertex is surrounded by two equilateral triangles, a square, another equilateral triangleand finally a square.

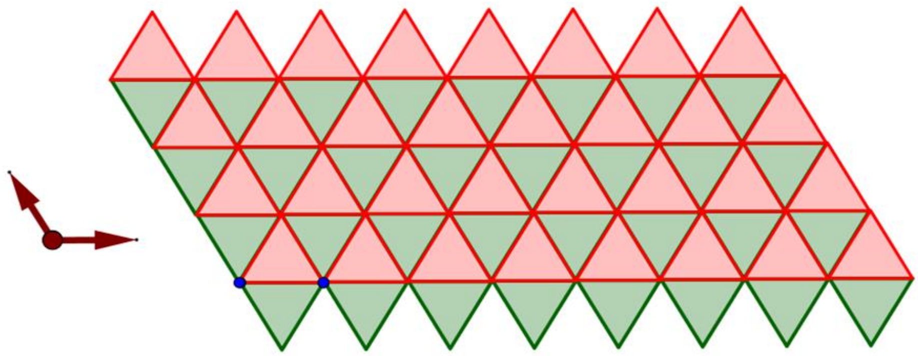
Clearly the vertex configuration of each vertex of a regular tessellations of the plane will be identical.

There are only three regular polygons that produce edge-edge regular tessellations of the Cartesian Plane:

2.1 Equilateral triangles tessellate the plane:

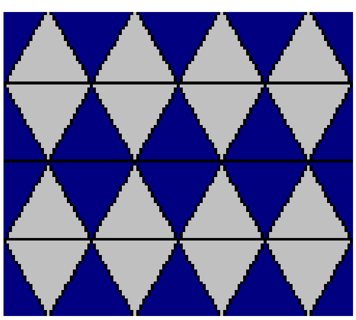


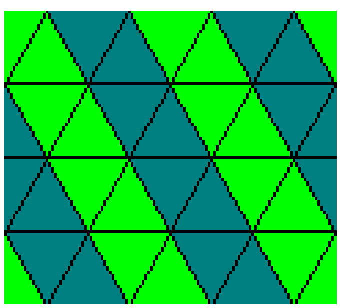
- Use the  tools to construct an equilateral triangle.
- Select triangle and use Copy and Paste functions to create a few copies in the Graphics View area.
- Drag and rotate around a common vertex to show that a regular polynomial of order 3 (equilateral triangle) tessellates the plane.

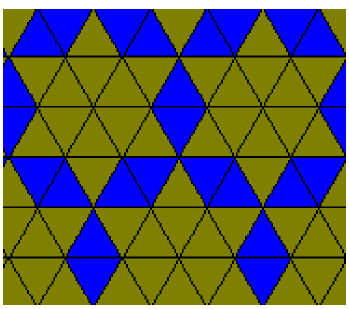


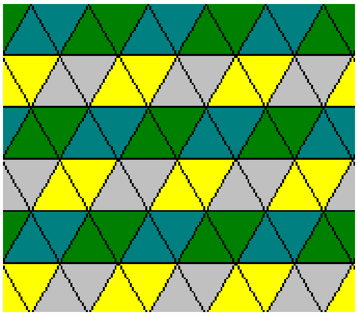
Note: The vertex configuration of each point of the tessellation is identical as 6 equilateral triangles surround each vertex. Hence this edge-edge regular tessellation could be described as a **3.3.3.3.3.3 - tiling** of the plane.

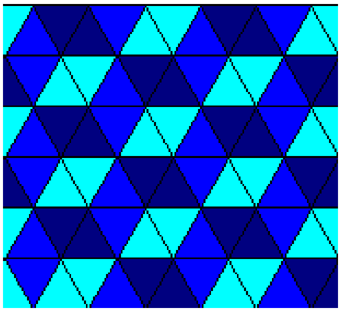
Some color pattern variations with regular tiling by equilateral trangles:

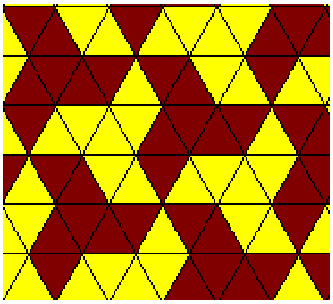




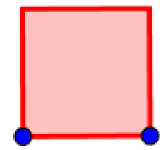


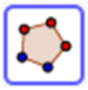


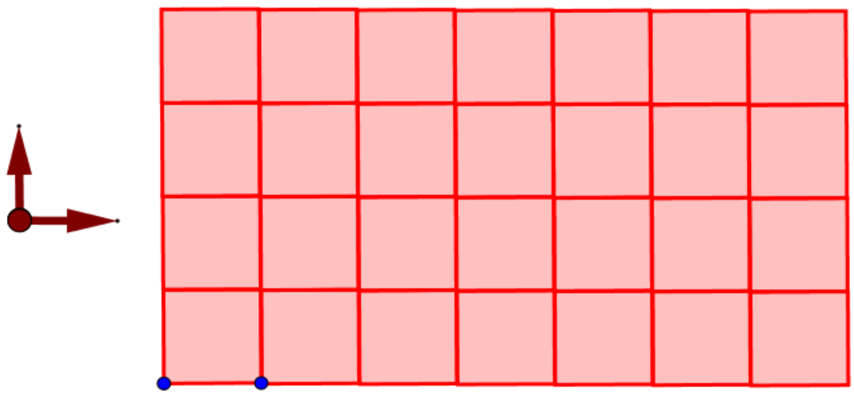




2.2 Squares tessellate the plane:

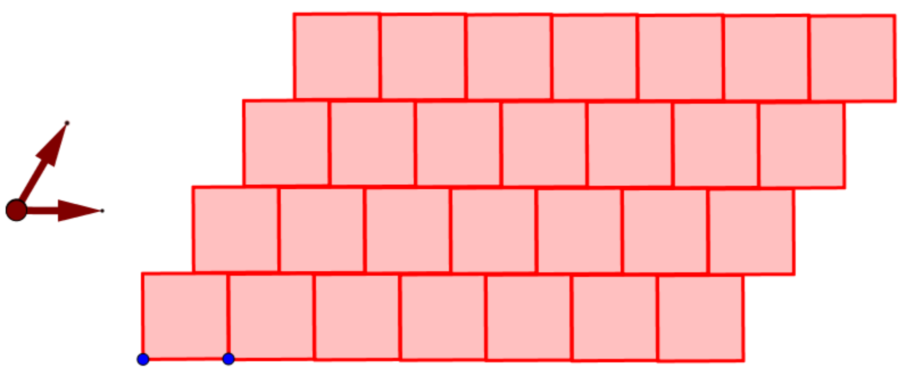


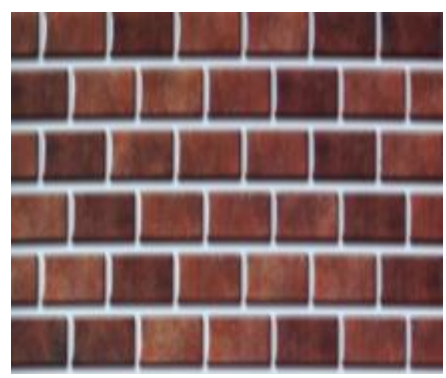
- Use the  tools to construct a square.
- Select the square and use Copy and Paste functions to create a few copies in the Graphics View area.
- Drag and rotate around a common vertex to show that a regular polynomial of order 4 (square) tessellates the plane.



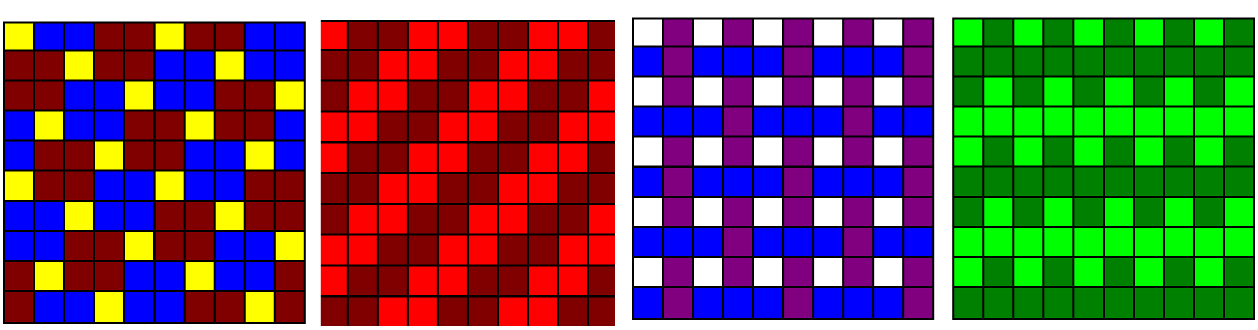
Note: The vertex configuration of each point of the tessellation is identical as 4 squares surround each vertex. Hence this edge-edge regular tessellation could be described as a **4.4.4.4 - tiling** of the plane.

Also note that there are infinite number of ways that a square could tesslate the plane. This can be done by sliding each horizontal layer in relation to the one above or below it. Below we show a tiling of the plane which frequently appears in bricklaying.

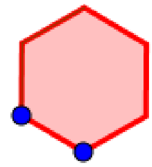


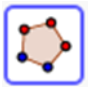


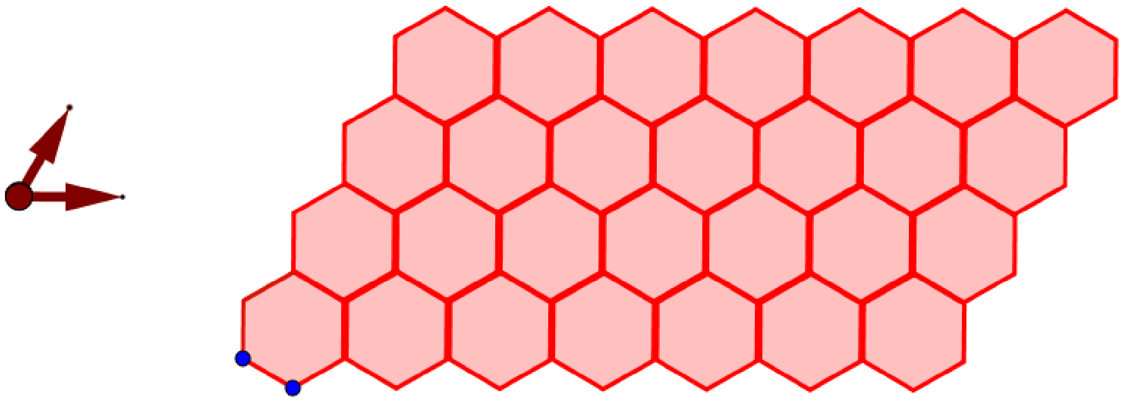
Some color pattern variations with regular tiling by squares:



2.3 Hexagons tessellate the plane:



- Use the  tools to construct a regular hexagon.
- Select the hexagon and use Copy and Paste functions to create a few copies in the Graphics View area.
- Drag and rotate around a common vertex to show that a regular polynomial of order 6 (hexagon) tessellates the plane.

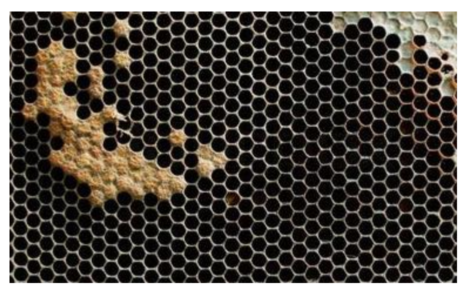


Note: The vertex configuration of each point of the tessellation is identical as 3 regular hexagons surround each vertex. Hence this edge-edge regular tessellation with hexagons could be described as a **6.6.6 - tiling** of the plane.

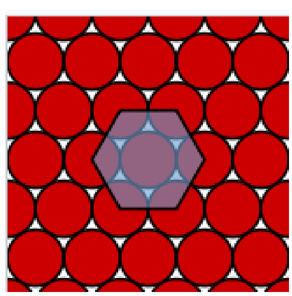
Some natural hexagonal packing structuresin a plane:

- Honey bees and their hives.

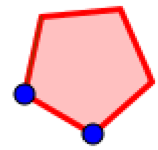


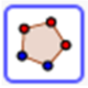


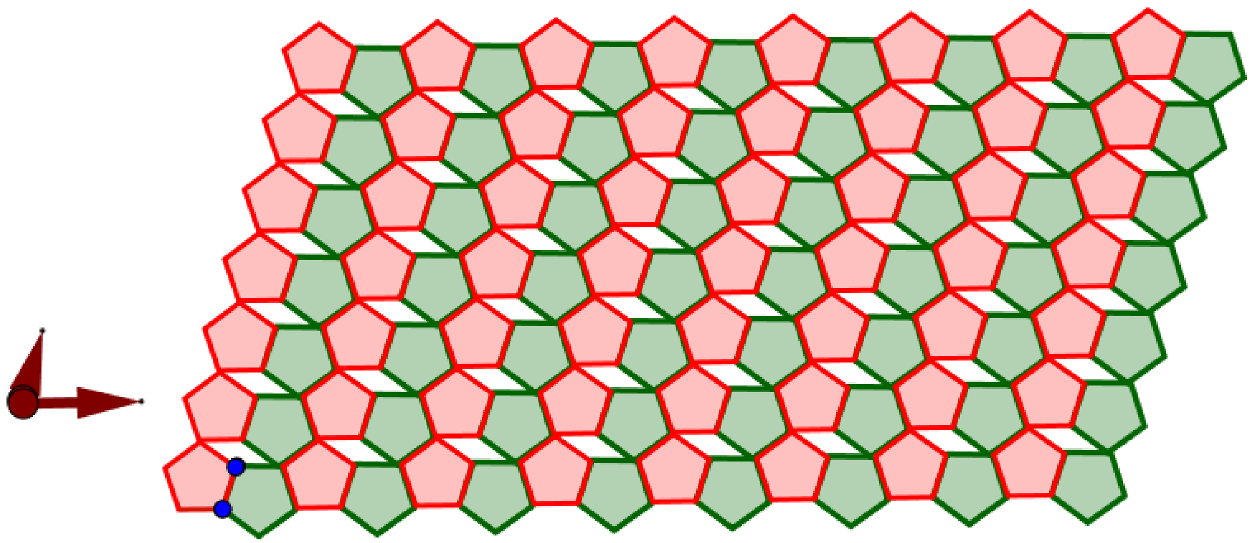
- Identical circles in a hexagonal packing arrangement, the densest “best” packing possible.



3. A Pentagon or any Regular Polygons of order >6 does not tessellate the plane:



- Use the  tools to construct a regular pentagon.
- Select the pentagon and use Copy and Paste functions to create a few copies in the Graphics View area.
- Drag and rotate around a common vertex to show that a regular polynomial of order 5 (pentagon) **does not** tessellates the plane.
- Pentagons does not fill the full cycle of $360°$ about each vertex.



Why not? The answer lie in the pattern of inner angles that are associated with regular polygons as reflected in Table 1.

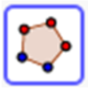
Order n of Regular Polygon	Common Name	Inner Angle	Factor of $360°$?
3	Equilateral Triangle	$\frac{(n-2)\times 180°}{n} = \frac{(3-2)\times 180°}{3} = 60°$	Yes
4	Square	$\frac{(n-2)\times 180°}{n} = \frac{(4-2)\times 180°}{4} = 90°$	Yes
5	Pentagon	$\frac{(n-2)\times 180°}{n} = \frac{(5-2)\times 180°}{5} = 108°$	No
6	Hexagon	$\frac{(n-2)\times 180°}{n} = \frac{(6-2)\times 180°}{6} = 120°$	Yes
7	Heptagon	$\frac{(n-2)\times 180°}{n} = \frac{(7-2)\times 180°}{7} = 128.57°$	No

8	Octagon	$\frac{(n-2)\times 180^\circ}{n}=\frac{(8-2)\times 180^\circ}{8}=135^\circ$	No
9	Nonagon	$\frac{(n-2)\times 180^\circ}{n}=\frac{(9-2)\times 180^\circ}{9}=140^\circ$	No

For a regular polygon to poduce an edg-edge tessellation of the plane, **the inner angle of the polygon must be a factor of** 360°.

Hence only equilateral triangles, squares and hexagons will tesselate a plane.

Exercise 1:

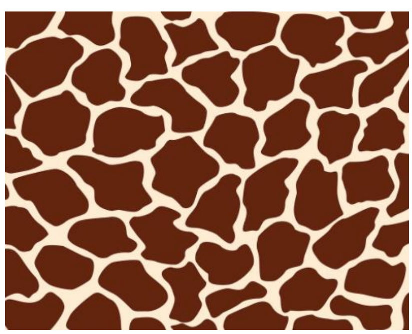
- Use the  tools to construct a regular pentagon with order >6.
- Select the polygon and use Copy and Paste functions to create a few copies in the Graphics View area.
- Drag and rotate polygons around a common vertex to show that a regular polynomial of order greater than 6 **does not** tessellates the plane.

Exercise 2: Discuss and explore the basic symmetrie of the Regular Tessellations of a plane.

Some natural examples of Irregular Tessellations of surfaces:

Scales on skin of reptiles

Animal Skin – Girrafe



Dried mud of water pool

Many others !!!!!