

Digging deeper using GeoGebra: An exploration of quadratics and more.

Abstract

Using GeoGebra students can explore in far more depth topics that have until recently been given a 'standard' treatment. One example is the exploration of the vertex of a quadratic function. By exploring how the vertex moves when the coefficients of the quadratic function change, various topics can be introduced. Some of these are specific to GeoGebra—dynamic graphing and sliders, for example. But we can also introduce students to parametric equations, eliminating the parameter and even graphing functions of two variables in 3-space. Other examples for possible exploration are given at the end.

Introduction

"Teachers should have experiences using technology as a computational and problem solving tool. When technology is used as a computational tool, learners use it to perform a calculation or produce a graph or table in order to use the result as input to analyze a mathematical situation. *They should also learn to use technology as a problem solving tool, or to conduct an investigation by taking a deliberate mathematical action, observing the consequences, and reflecting on the mathematical implications of the consequences.*" (Bold and italics added.) (CBMS MET II, 2012, pp. 33-34)¹

That bold faced part of the quote above will be illustrated by an in-depth exploration of the vertex of a quadratic equation, $f(x) = ax^2 + bx + c$.

Exploration

Students can begin graphing various specific quadratic functions such as $f(x) = 2x^2 + 3x - 4$ or $g(x) = -x^2 + 2x + 3$. (See Figures 1a, 1b, 1c below.) We can then add sliders for the coefficients a , b and c in $f(x) = ax^2 + bx + c$ to make the graphing dynamic

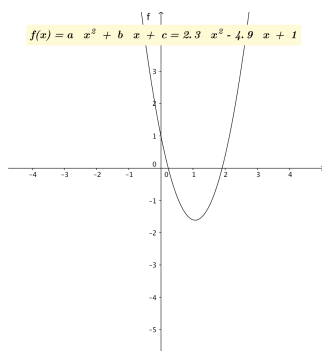


Fig. 1a

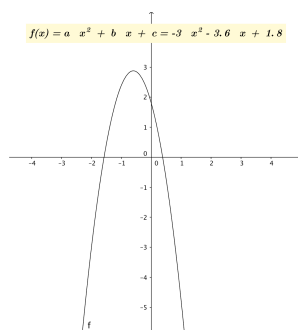


Fig. 1b

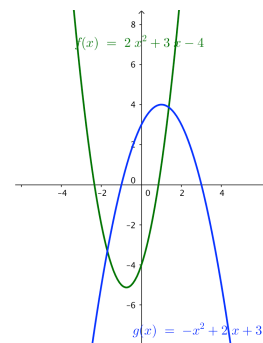
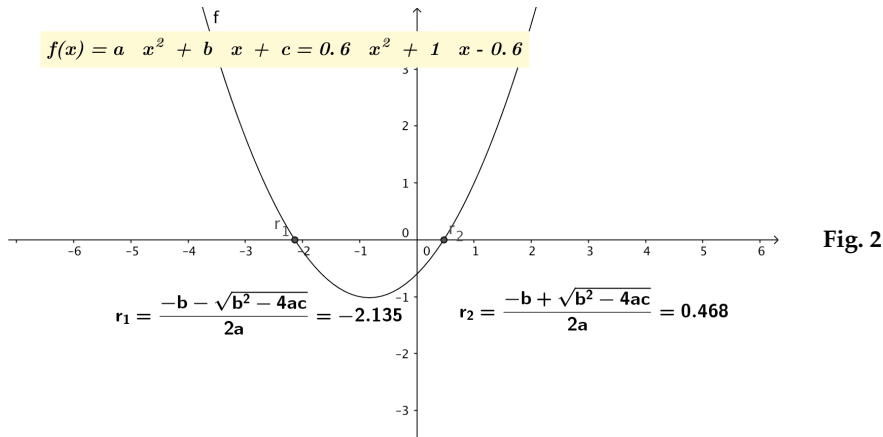


Fig. 1c

Let's explore finding axis of symmetry and the vertex. One way to find the axis of symmetry is first find the solutions to $f(x) = 0$ and then finding the midpoint of the x -coordinates of the roots. (Fig. 2)



The x -coordinate of the vertex is easily seen to be $-\frac{b}{2a}$. But what happens when there are no roots? The formulaic manipulation gives the same answer, i.e. average the two complex roots to find $-\frac{b}{2a}$ but there is no (easy) graphical evidence to support this. There is a way around this. The roots, when they exist, are just the intersection of the quadratic with the horizontal line $y = 0$ and these points of intersection will be the same distance on either side of the axis of symmetry. But this property holds for *any* horizontal line that intersects the graph. There is one horizontal line that is guaranteed to intersect the graph: $y = c$. (Fig. 3a, 3b)

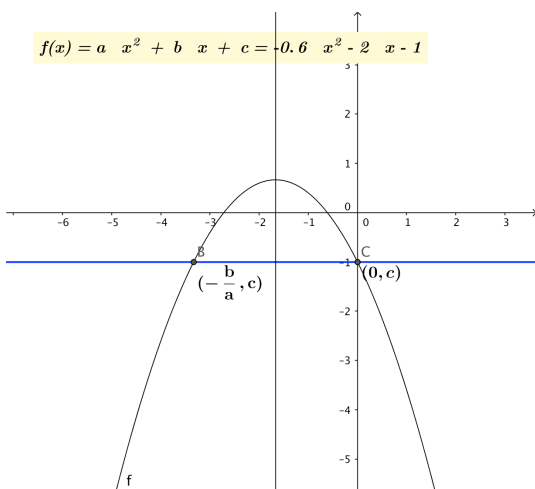


Fig. 3a

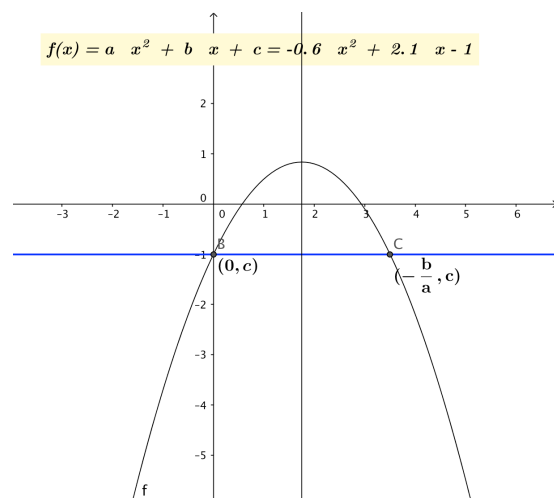


Fig. 3b

GeoGebra does not replace algebra or paper and pencil work (although it can do some very good algebra). Setting $ax^2 + bx + c = c$ and solving gives $x(ax + b) = 0$ so $x = 0$ or $x = -\frac{b}{a}$. The midpoint of these values is $x = -\frac{b}{2a}$. Note that $x = 0$ is always a solution. If $b = 0$ then in this case the y -axis is the axis of symmetry.

Sliders & Dynamic Graphing

With sliders the students make graphing quadratics dynamic. But there is more one can do with GeoGebra. Add the vertex and then trace the vertex while changing one of the parameters while keeping the other two fixed. (Figures 4a, 4b, 4c)

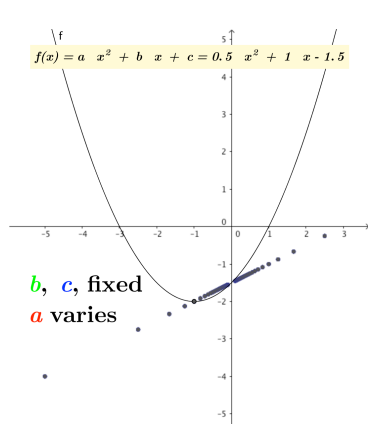


Fig. 4a

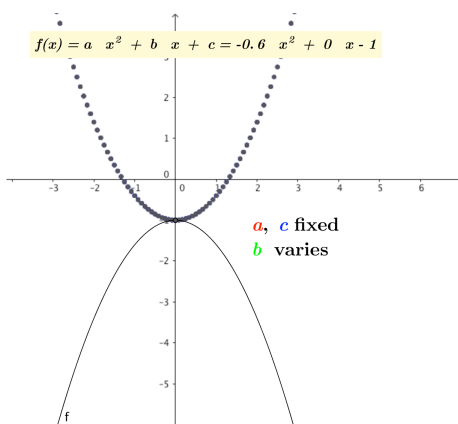


Fig. 4b

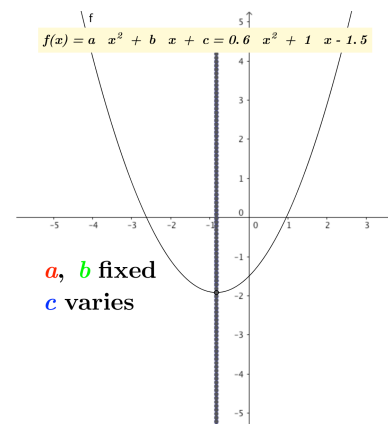


Fig. 4c

By now they should know (or have a CAS help them find) the vertex $V = \left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right)$. We

can use this to see why changing c and keeping a, b fixed just moves the vertex up or down. (Figure 4c)

Parametric Equations

Changing a and b give more interesting changes. We can use these changes to introduce parametric equations. Keeping a and c fixed and allowing b to range over all reals we could

write

$$x = -\frac{1}{2a}b$$

$$y = -\frac{1}{4a}b^2 + c$$

Let $t = b$ and then write $x = -\frac{1}{2a}t$. Now eliminate the parameter by substituting $t = -2ax$
 $y = -\frac{1}{4a}t^2 + c$

to get $y = -ax^2 + c$. This is just the original quadratic with $b = 0$ reflected over the line $y = c$ as Figure 4b above illustrates.

The last case, moving only a , is also interesting and can be shown to be a line but now we

would have $t = \frac{1}{a}$. The parametric equations are $x = -\frac{b}{2}t$. This is illustrated in Figure 4a.
 $y = -\frac{b^2}{4}t + c$

An interesting discussion could take place about using $t = \frac{1}{a}$. Too often we (i.e teachers) do not point out that this transformation maps the interval $(0,1]$ to $[1,\infty)$ and the interval $[1,\infty)$ to $(0,1]$. That is why when a gets large (either positive or negative) the vertex moves to the point $(0,c)$ and as a approaches 0, the vertex shoots off to infinity.

Eliminating the Parameter

A student might then ask how those traces change when you change one of the fixed parameters. Here the locus command is useful since they will move when b is changed. The red lines in Figures 5a, 5b and 5c below are all generated by the locus command.

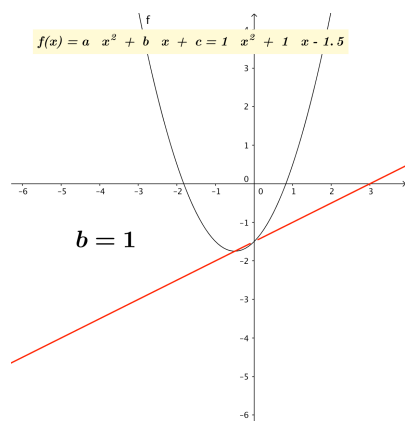


Fig. 5a

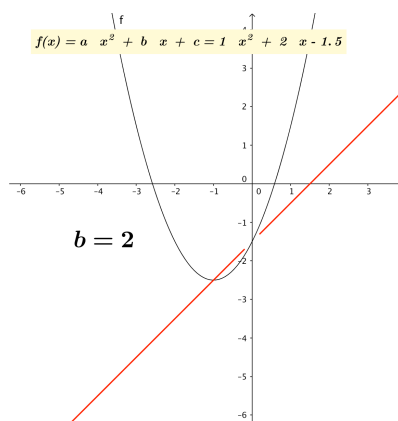


Fig. 5b

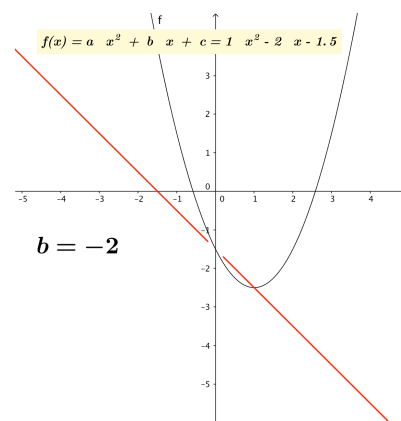


Fig. 5c

The slopes of these lines clearly depend on b . If we eliminate the parameter we get $y = \frac{b}{2}x + c$ so the slope of the a -locus is $\frac{b}{2}$. Note that these lines appear to always go through the point $(0, c)$. However, that point is not really on the line since it would happen only if $\frac{1}{a} = 0$.

Similarly we see the effect that a has on the b -locus. Since the b -locus is just a mirror image of the original quadratic with $b = 0$, changing a has the same effect on both the quadratic and the b -locus.

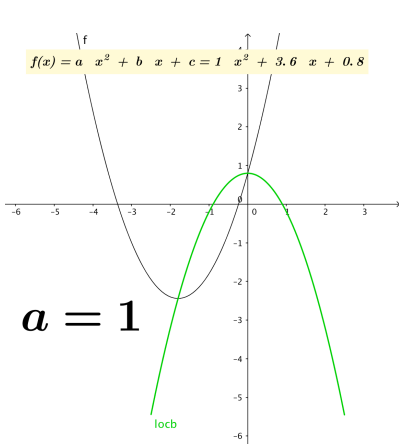


Fig. 6a

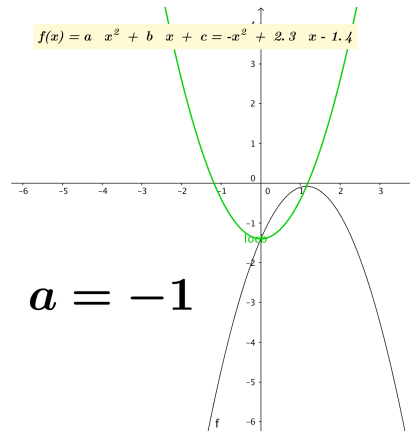


Fig. 6b

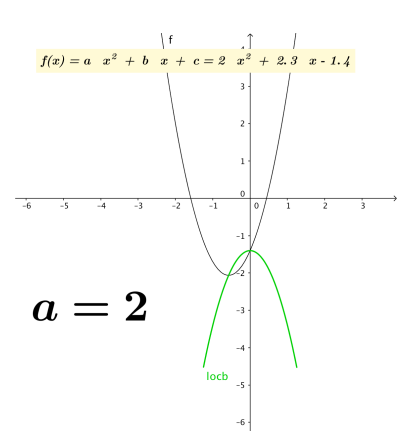


Fig. 6c

3-D graphing

One way to visualize the movement of the vertex is to move to 3 dimensions. If we 'stacked up' the paths of the vertex as the fixed parameter changes, we would generate a surface.

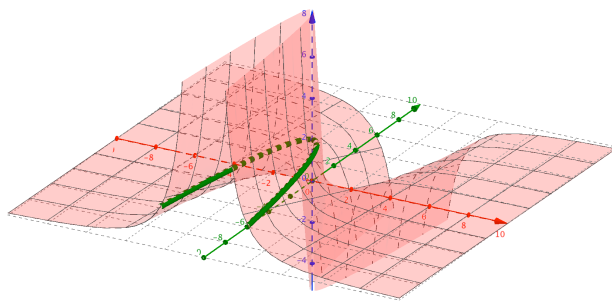


Fig. 7a

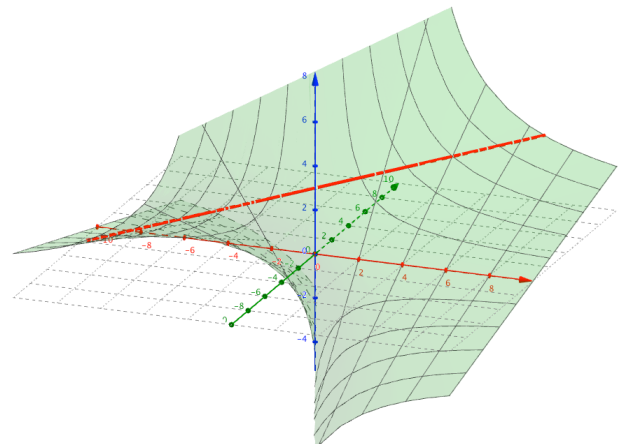


Fig. 7b

To make things a bit simpler, I will now let $c = 0$ for the rest of the discussion.

The surface in Fig. 7a is the image of $\left(-\frac{b}{2a}, -\frac{b^2}{4a}, a\right)$ where a and b range over all reals.. The green parabola is the path of the vertex for fixed $a = 2$.

The surface in Fig. 7b is $\left(-\frac{b}{2a}, -\frac{b^2}{4a}, b\right)$. The red line shows the vertex path for the fixed $b = 3$.

Both the parabola and line can be moved along the 'a' or 'b' axis by moving sliders.

Functions of two variables

These pictures can be used to introduce functions of two variables. If we let $x = -\frac{b}{2a}$, $y = -\frac{b^2}{4a}$ and $z = a$ then it can be shown that $z = -\frac{y}{x^2}$ so if we graph in 3D the function $f(x,y) = -\frac{y}{x^2}$ we

would find the same surface as in Fig. 7a above. Similarly, $f(x,y) = \frac{2y}{x}$ gives the surface in

Fig. 7b. The advantage of using functions of two variables rather than the Surface command is that it opens up more GGB power. One can intersect a plane and a function and GGB will find the equation of the intersection which is not possible with surfaces in GGB.

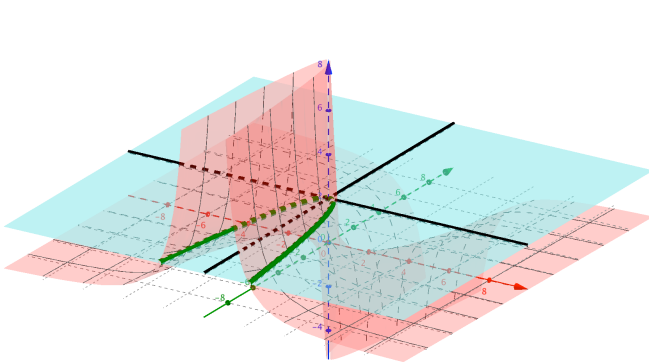


Fig. 8a

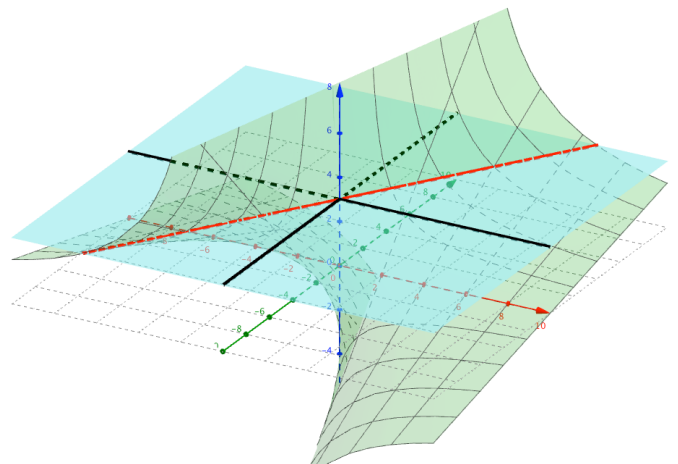


Fig. 8b

More examples of digging deeper:

- Making vectors dynamic to model some complex motions or to discover unusual patterns.
- Using the Surface, Rotate and Sequence commands to demonstrate volumes of revolution and of known cross sectional areas. These will be demonstrated as time permits.

Vectors

A simple example using dynamic vectors

Take two concentric circles of radius 1 and 2. Have vectors whose tails are at the center and whose heads are on the respective circles. Rotate the vectors at different angular speeds and plot the sum of the two vectors. Here are some paths the resultant vector takes with varying speeds.

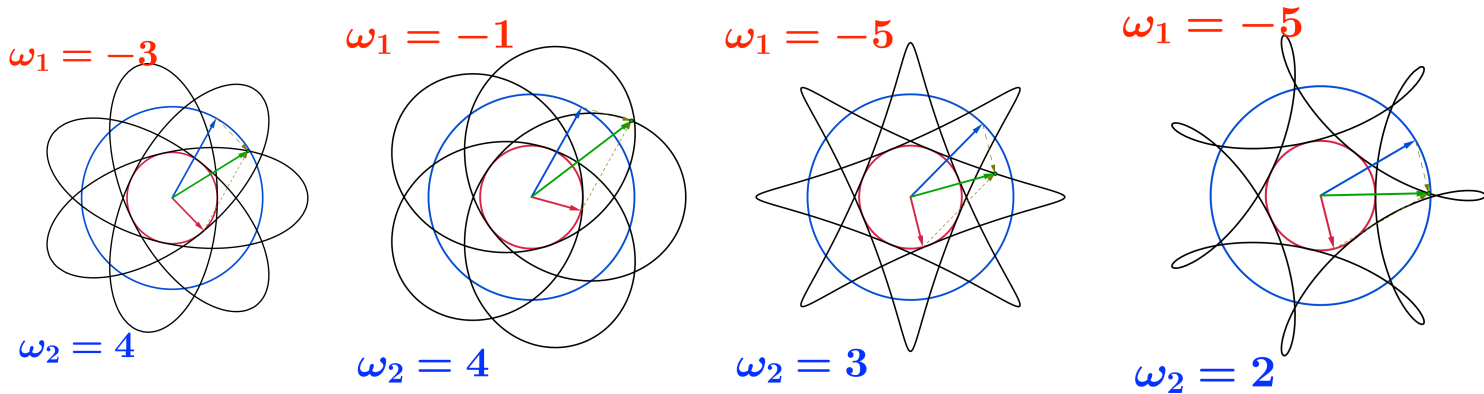


Fig. 9a

Fig. 9b

Fig. 9c

Fig. 9d

Students can then write the parametric equations of the above curves. For example, the parametric equations for the star-shaped third curve (Fig. 9c) are $x(t) = 2 \cos\left(\frac{2\pi}{3}t\right) + \cos\left(\frac{2\pi}{5}t\right)$ and $y(t) = 2 \sin\left(\frac{2\pi}{3}t\right) - \sin\left(\frac{2\pi}{5}t\right)$. In general the parametric curves are given by $x(t) = 2 \cos\left(\frac{2\pi}{\omega_2}t\right) + \cos\left(\frac{2\pi}{\omega_1}t\right)$ and $y(t) = 2 \sin\left(\frac{2\pi}{\omega_2}t\right) + \sin\left(\frac{2\pi}{\omega_1}t\right)$.

Mad Teacups (or mathematics at Disneyland) using vectors:

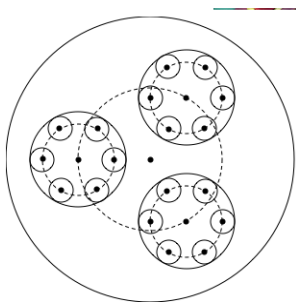


Fig. 10a

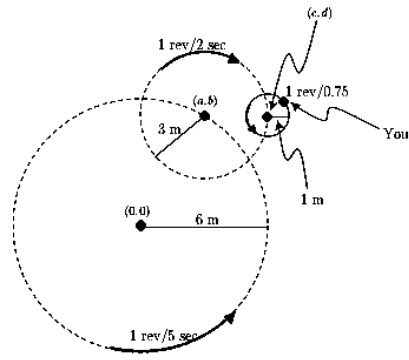


Fig. 10b

Rotating vectors can be used to model a Mad Tea Cups ride similar to the one found at Disneyland or Walt Disney World.

The first schematic (Fig. 10a) shows the overview: A large circular plate rotating counter-clockwise, three smaller circular plates rotating faster clockwise and on these plates are six 'tea cups' rotating the fastest counterclockwise. The second schematic shows the direction and revolutions/second for each of the rotating circles. The question is what does the path of a person sitting on a teacup look like? If a student know that a a point moving around a circle of radius R centered at (h,k) and taking p seconds to make 1 revolution, then the parametric equations are $x(t) = h + R \cos\left(\frac{2\pi}{p}t\right)$ and $y(t) = k + R \sin\left(\frac{2\pi}{p}t\right)$. The key is to recognize that (h,k) can be dynamic. That is, the center of a given circle could also be moving around another circle or path.

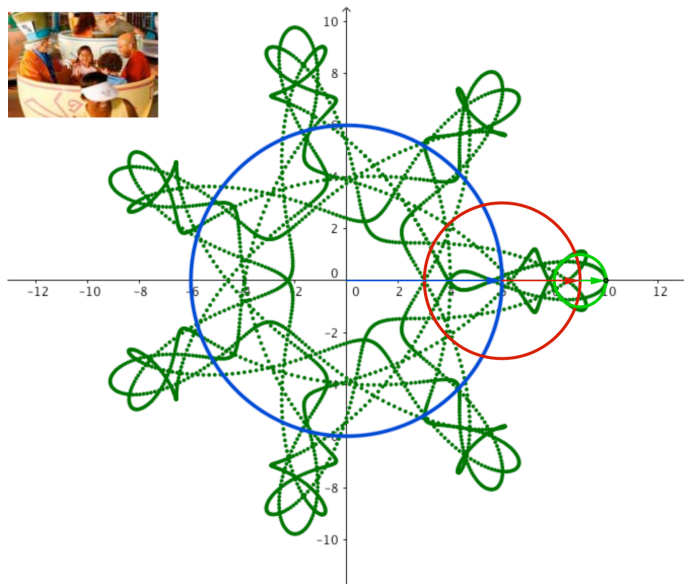


Fig. 11

The green trace in Fig. 11 shows the path of a passenger on a tea cup. The parametric equations are:

$$x(t) = 6 \cos\left(\frac{2\pi t}{5}\right) + 3 \cos\left(\frac{2\pi t}{2}\right) + \cos\left(\frac{2\pi t}{0.75}\right)$$

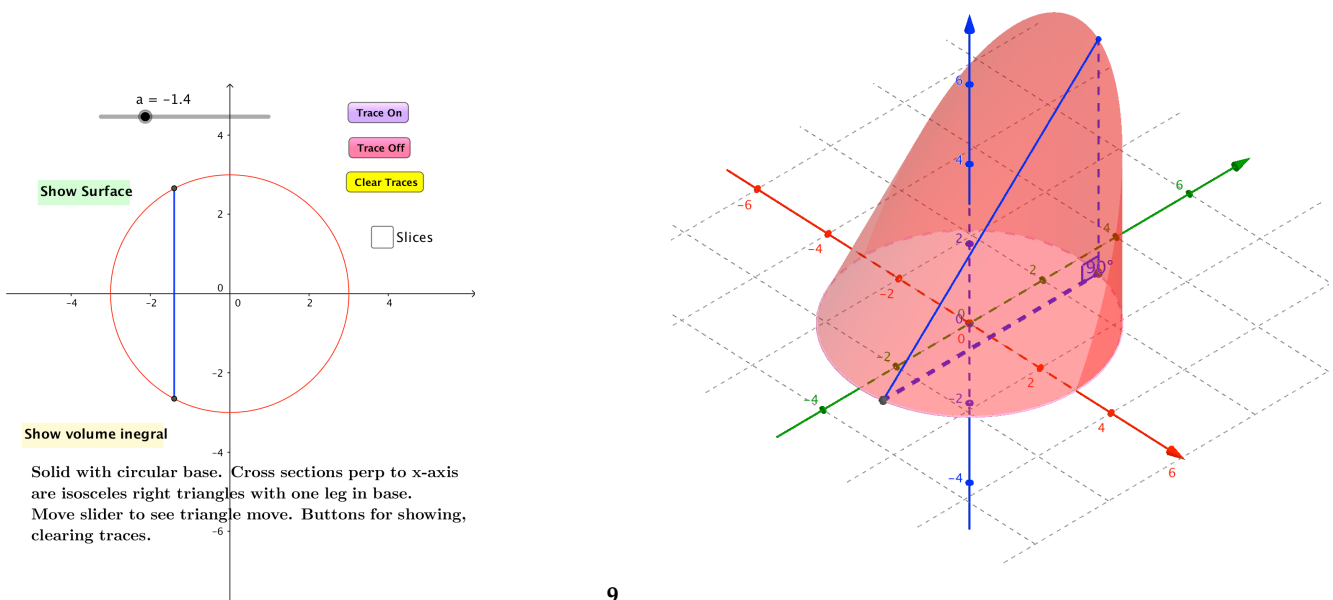
$$y(t) = 6 \sin\left(\frac{2\pi t}{5}\right) - 3 \sin\left(\frac{2\pi t}{2}\right) + \sin\left(\frac{2\pi t}{0.75}\right)$$

I should point out that I do not need these written this way to create the GeoGebra applet. Instead I defined $A = \left(6 \cos\left(\frac{2\pi t}{5}\right), 6 \sin\left(\frac{2\pi t}{5}\right)\right)$, $B = A + \left(2 \cos\left(\frac{2\pi t}{2}\right), -2 \sin\left(\frac{2\pi t}{2}\right)\right)$ and finally $C = B + \left(\cos\left(\frac{2\pi t}{0.75}\right), \sin\left(\frac{2\pi t}{0.75}\right)\right)$. The green path above shows how point C moves. Points A and B are dynamic centers. Students can look for regularities within the path, i.e. look beyond the initial chaotic impression.

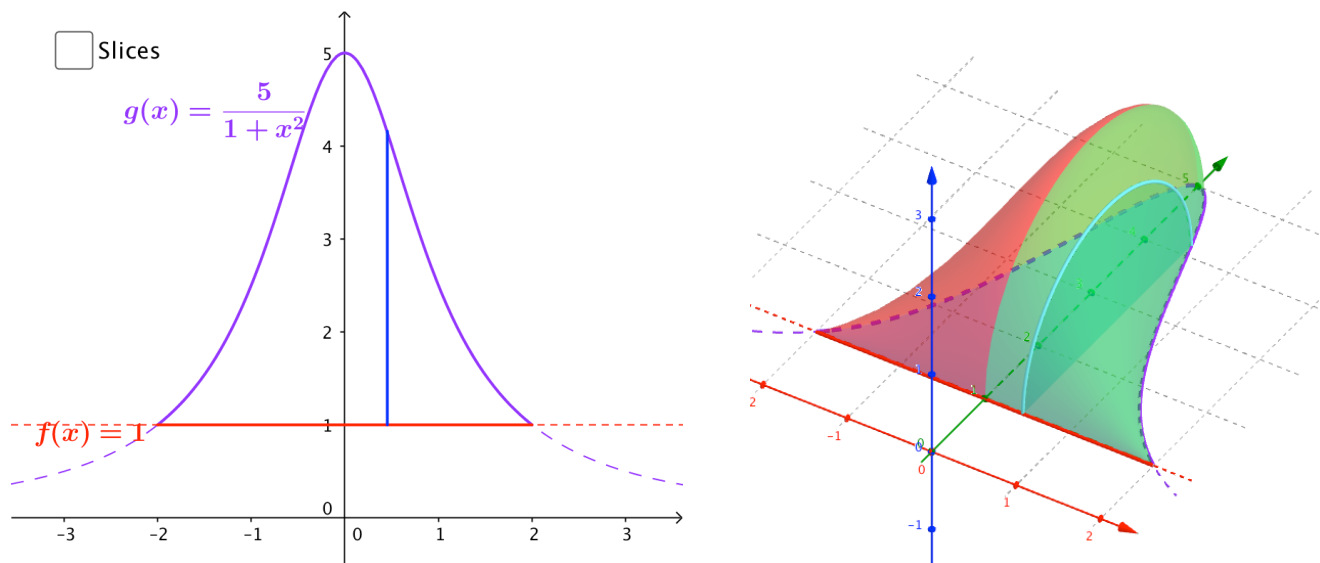
Volumes in Calculus

Students are often asked to find the volumes of solids with known cross sectional areas. Often they can find the volume without really knowing what the solid really looks like. Below are examples using the 3D capabilities of GeoGebra to show what some of these solids look like.

Circle base, cross sections perpendicular to diameter are isosceles right triangles with one leg in base.



Base is region between $g(x) = \frac{5}{1+x^2}$ and $f(x) = 1$. Cross section slices perpendicular to the x -axis are semicircles with diameter in base.



Summary

In the previous examples we tried to illustrate the CBMS recommendations to take a deliberate mathematical action, observe the consequences and reflect on the implications. With the vertex of a quadratic function, we asked what would happen to the vertex if we changed one of the coefficients, discovering lines and a new parabola. Next, we asked how those lines and parabolas change when we change the other parameters. Finally, we wondered what the images would look like if the loci were stacked up to create a 3D surface.

For the vector examples, conjectures included asking what would happen if rotated the vectors at varying speeds or made them move on circles whose centers were moving also. For volumes, we could ask what changes when the cross sections are semicircles rather instead of squares or triangles. GeoGebra allows us to see the different solids and not just calculate the volumes. GeoGebra allows us to explore topics much deeper than we have

¹ Conference Board of the Mathematical Sciences (2012).

The Mathematical Education of Teachers II.

American Mathematical Society and Mathematical Association of America.

<http://cbmsweb.org/MET2/>

All of the GeoGebra applets used can be found at dougkuhlmann.webs.com under the heading SoCTggbConf or at tube.geogebra.org.

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