Polygons

1. *Definition* – How do we define polygons? Is there an only way to define polygons?
   a. Discuss the following students' definitions of polygons. Are they valid? How would you respond to them?
      i. Polygon is a simple closed shape that contains straight lines.
      ii. Polygon is a union of several line segments that form angles.
      iii. Polygon is several line segments meeting at their endpoints.

1. Definition - Poly- many Gon- angles. A simple closed curve exclusively composed of line segments. There are many ways to define polygons. A lot of times the definition depends on the author who writes them.
   a. Are they valid?
      i. Polygon is a simple closed shape that contains only line segments.
      ii. Polygon is a union of several line segments that are closed and form angles
      iii. Polygon is several finite line segments that have exactly one finite line segment attached to each endpoint.

2. *Vocabulary*: Side, Vertex, Diagonal, Interior angle, Exterior angle; Regular polygon;
   a. Discuss the differences in the definition of an interior and exterior angle. Which one is less intuitive and why?
   b. Define the diagonal of a polygon and discuss the following students' claims:
      i. Diagonal is the longest line segment that can be drawn in a polygon.
      ii. A diagonal always cuts the polygon in two parts.
      iii. How to find the number of diagonals in a polygon: N-sided polygon has n vertices. N vertices can form (n-1) + (n-2) + (n-3) + ... + 1 + 0 pairs, which is the same as the number of diagonals. (Explain the student's thinking and state your response).

2. Vocabulary
   a. Side- A line segment connected to form a polygon.
   b. Vertex- A point where two or more line segments meet.
   c. Diagonal- The linking of two non-adjacent sides.
   d. Interior angle- Angles inside a polygon
   e. Exterior angle-The Exterior Angle is the angle between any side of a shape, and a line extended from the next side. (will always form a 180° angle)

a) The interior angle is less intuitive because it requires less work to find.
b) Define the diagonal of a polygon & discuss students claims:

    i)   Diagonal is longest line segment that can be drawn in a polygon.
        The diagonal doesn't necessarily have to be drawn inside a polygon
        (concave polygons).
        For a triangle there are no diagonals.

    ii)   Diagonal always cuts the polygon in two parts
        True for every polygon except triangles and some diagonals within concave
        polygons.
    iii)   Pairs of vertices are not necessarily diagonals and they are counting
   them as such.
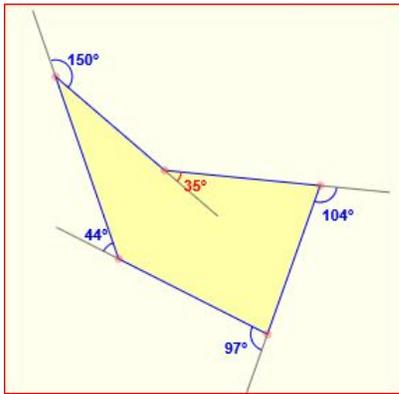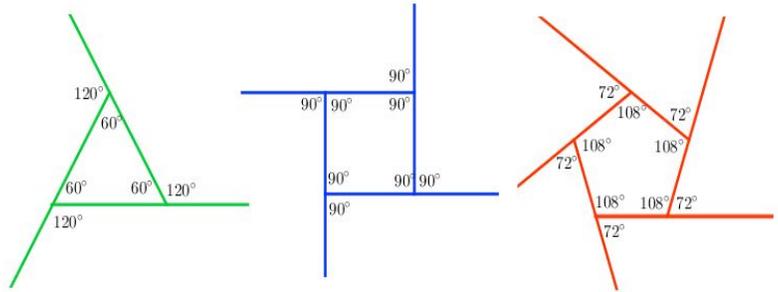
Concave   Convex

    f.   Regular polygon- A polygon that is equiangular and equilateral.

3. *Convexity and concavity.* What is a convex/non-convex figure? How would you define a convex shape using mathematical language? (Note: the question refers to figures in general, not necessarily polygons.

    a.   A student describes a convex shape as a shape, in which all interior angles are less than 180°. Is it correct? How would you respond?
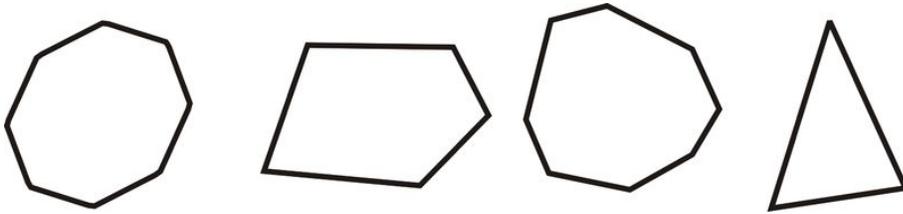
3. **Answer: The student is correct.**

Concave
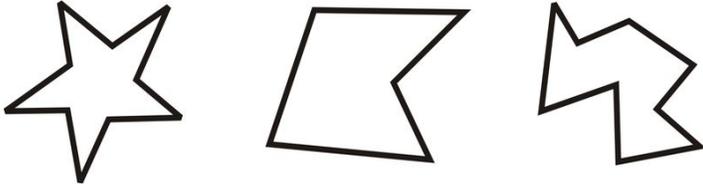
A concave polygon is defined as a polygon with one or more interior angles greater than 180°.

Convex

A convex polygon is defined as a polygon with all its interior angles less than 180°

4. *Classification of polygons.* Review the names of polygons (up to decagon and including dodecagon). What is a regular polygon?

   a. How would you respond to the following two statements:

      i. Regular polygon is a polygon with all sides congruent.

      ii. Regular polygon is a polygon with all angles congruent.

Regular polygon= polygon with all equal angles and sides

4. Answer
   a. Triangle 3
   b. Quadrilateral 4
   c. Pentagon 5
   d. Hexagon 6
   e. Heptagon 7
   f. Octagon 8
   g. Nonagon 9
   h. Decagon 10
   i. Dodecagon 12

i) Statement one is not correct, consider a Rhombus (all sides are congruent but not all of the angles are. Therefore, it is not a regular polygon.
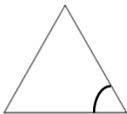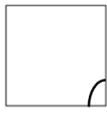
ii) Statement two is not always true. For example a rectangle. All angles are congruent but not all sides are congruent

5. *Interior angles sum.*
   i. Suggest several ways (deductive and inductive) to derive a formula for the sum of interior angles of an n-sided polygon.

   ii. Derive a formula for the interior angle of an n-sided regular polygon.
   iii. Find the interior angle of a regular pentagon, octagon, … etc.

5. Answer:
   180(n-2)/n where n is the number of sides

Regular Polygon with Interior Angles		www.cazoommaths.com
Shape	**Interior Angle**	**Name**
	$60°$	Equilateral Triangle
	$90°$	Square
	$108°$	Pentagon
	$120°$	Hexagon
	$128.57°$	Heptagon
	$135°$	Octagon
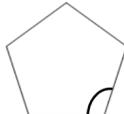
1. BASIC IDEA:
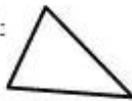
The interior angles of any triangle add up to 180 degrees

2.
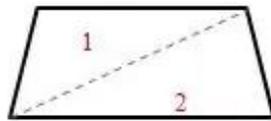
This is a Tr.......

It would split into ⬜ triangles

So the interior angles add up to 2 × ⬜ = 360⁰

3.
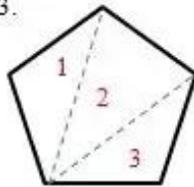
This is a REGULAR Pent........

It would split into ⬜ triangles

So the interior angles add up to 3 × ⬜ = 540⁰

So *each interior* angle = 720 / 5 = ⬜

4.
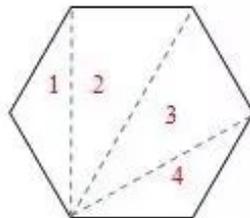
This is a REGULAR Hexa......

It would split into ⬜ triangles

So the interior angles add up to 4 × ⬜ = 720⁰

So each interior angle = 720 / ⬜ = ⬜

5.
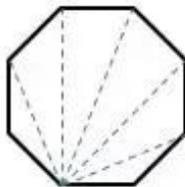
This is a REGULAR Octa......

It would split into ⬜ triangles

So the interior angles add up to 6 × ⬜ = 1080⁰

So each interior angle = 720 / ⬜ = ⬜

a. Inductive- Specific case to a general case
   Give students several different polygons with their angle measures and have them.
b. Deductive- General to specific
   Give students the formula and have them apply it to different polygons and note their observations.

n-sided

$$\frac{180n - 360}{180(n-2)}$$
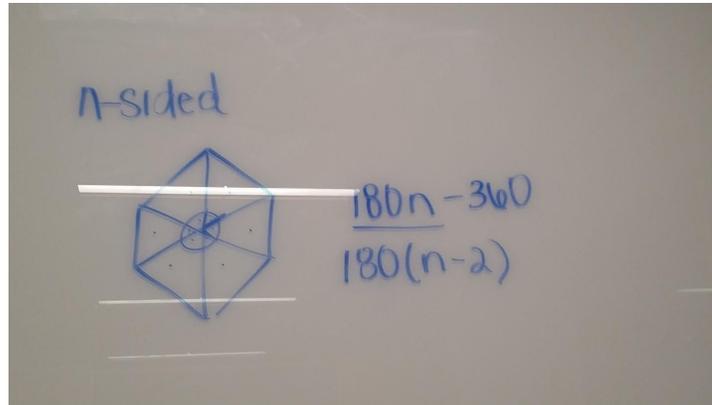
6. *Exterior angles sum.*

    i. Derive the formula for the exterior angles sum of an n-sided polygon using

        1. A visual method. (A "car" method" or "shrinking polygon" method – recall GeoGebra Tube worksheets we discussed)

        2. The interior angle sum formula and algebra.

    ii. The sum of all interior angles is the same as the sum of all exterior angles for some polygons. How would you respond to that?

    b. Equilateral means "equal sides". Equiangular means "equal angles". Respond to the following claims:

        i. A regular polygon is always equilateral.

        ii. An equilateral polygon is also equiangular.

        iii. An equiangular polygon is also equilateral.

        iv. A pentagon that has all sides congruent is a regular pentagon.

6. **Answer:**

i) 1. Marcinek geogebra "car method" activity link:

https://www.geogebra.org/m/uGD2N9Fw

2. All exterior angles will always add up to 360 then subtract concave angle, this will always add up to 360 degrees

ii) It is true only for 4 sided polygons

b)

i) True

ii) True

iii) False (Rectangle)

iv) True

Ex)
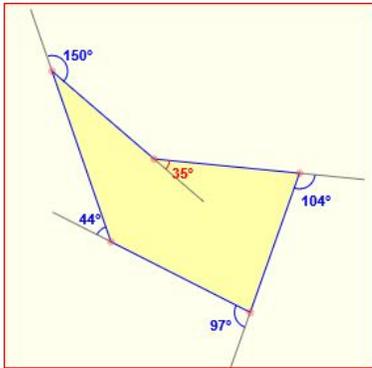
Add blue (exterior) and subtract red (concave angle) = 360 degrees

Consider a pentagon ABCDE.
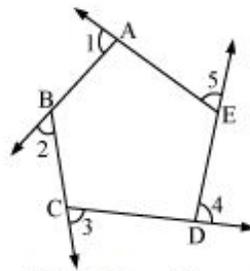
Sum of exterior angles of pentagon ABCDE = ∠1 + ∠2 + ∠3 + ∠4 + ∠5

∠1 = 180° − ∠BAE	(Linear pair of angles)
∠2 = 180° − ∠CBA	(Linear pair of angles)
∠3 = 180° − ∠DCB	(Linear pair of angles)
∠4 = 180° − ∠EDC	(Linear pair of angles)
∠5 = 180° − ∠AED	(Linear pair of angles)

∴ ∠1 + ∠2 + ∠3 + ∠4 + ∠5 = 900° − (∠BAE + ∠CBA + ∠DCB + ∠EDC + ∠AED)

∠BAE + ∠CBA + ∠DCB + ∠EDC + ∠AED
= Sum of interior angles of he pentagon ABCDE
= (5 − 2) × 180°   (Sum of interior angle of polygon of n sides = $(n - 2) × 180°$)
= 540°

∴ Sum of the exterior angles of pentagon = 900° − 540° = 360°

Similarly, we can prove that the sum of exterior angles of any polygon is 360°.

Defining and constructing specific polygons

You should be able to construct other objects (with made-up names) if given their minimal definitions.

    a.   "House": A pentagon with two adjacent right angles.

            a. "House": A pentagon with two adjacent right angles.

                    a.   Q1: Can a "house" have two diagonals that are perpendicular to each other?
                    b.   Q2: Can a "house" be a concave polygon?
                    c.   Q3: Is a regular pentagon a house?

    b.   "Funhex": A hexagon with all pairs of opposite sides congruent.

                    a.   Q1: Is a regular hexagon a Funhex?
                    b.   Q2: Can Funhex be a concave polygon?

No proof is necessary, just make sure the answers can be seen from your picture.

Defining and constructing specific polygons

House

a) If the three sides adjacent to the 90 degree angles are the same length

b) yes it can be a concave polygon if you move the top triangle into the rectangle part
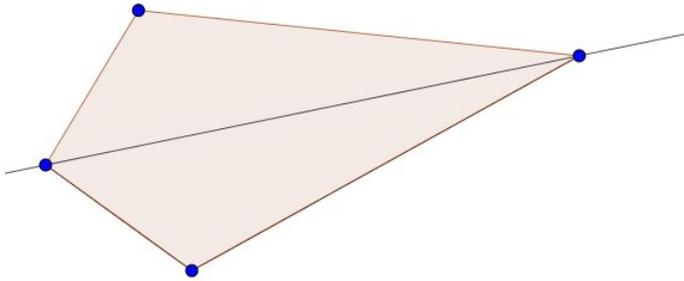
c) No, a regular pentagon must have all equal sides and angles, therefore there would not be 90 degree base angles.
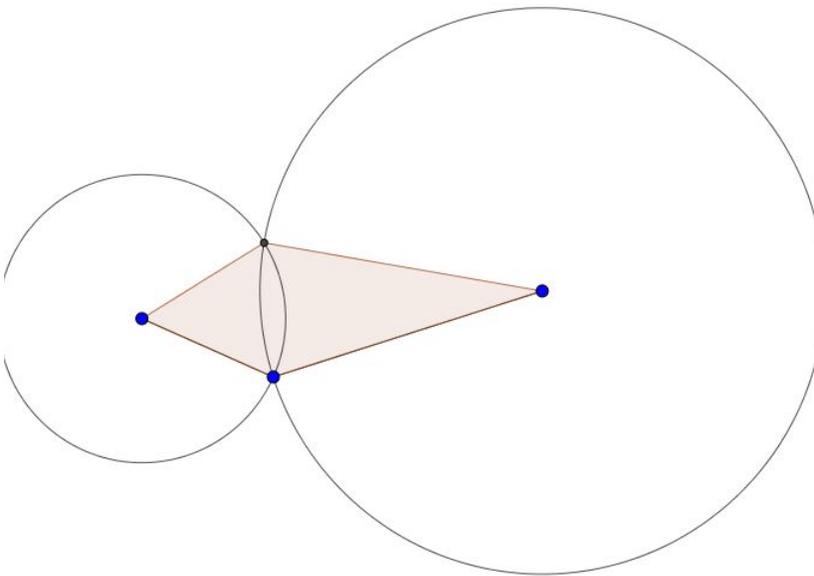
Funhex

a) Yes it is because all sides are congruent so the opposite sides would be congruent.

b) Yes it can be a concave polygon

Quadrilaterals

Properties of Quadrilaterals

You should be able to construct quadrilaterals given their minimal definitions (you do not have to memorize the definitions, they will be provided). Make sure your object is not over- or under-constrained.

You may use your construction to answer questions about the shape's properties (refer to our chart). Keep in mind that these properties must hold for ALL quadrilateral included in the category, not just the

one displayed on your screen. If a property does not hold for all given quadrilaterals, make sure you provide a counterexample. If it holds, briefly justify your answer by only assuming things that are to the _left_ of the box you are working on. How would your justification change if you were allowed to assume all the statements to the left and _above_ the box you are working on?

Below are just examples of possible questions. Make sure you are comfortable with all properties we discussed in class.

Over constrained: can't get every scenario out of it

Under constrained: you get too many different outcomes from it

Below are just examples of possible questions. Make sure you are comfortable with all properties we discussed in class.

1. There are several ways to define a quadrilateral which leads to several different ways to construct a quadrilateral (and the other way - different ways to construct the same quadrilateral imply different definitions of the quadrilateral). Construct a kite in three significantly different ways. Hint - think of different ways to define a kite:
    a. At least two pairs of distinct adjacent congruent sides
    b. One diagonal is a perpendicular bisector of the other diagonal.
    c. A quadrilateral symmetrical about at least one of its diagonals.

1. **Answer:**

https://www.geogebra.org/m/Mwa8uXhU

a) **Create a line segment - compass the line from one point to the other - place another point on the created circle - create angle bisector of the three points - then place another point on the line created by the angle bisector and connect the points with line segments in order to create a kite**

b) **Create two line segments attached to each other, put a line through the other two points. Click the point and reflect about the line.**

c) **Create two line segments. Compass each of them, mark a point where the circles intersect, and connect using polygon tool. (overconstrained)**

2. Kite: a quadrilateral with two distinct pairs of adjacent congruent sides.
    a. Our minimal definition of a kite involves the word *distinct* ("distinct pairs of adjacent congruent sides"). Explain what it means and why it is necessary there.
    b. Q: Are the diagonals always perpendicular?
       Q2: What other observation about kite's diagonals can you make?
    c. Justify the formula for the area of a kite (diagonals are known).

2. **Answer:**
a) **There are 2 distinct pairs of adjacent congruent sides, the pairs cannot have a side in common. If you take one pair out you only have one pair left**
b) **The diagonals are always perpendicular due to the kite being a reflection - the created line from one point to the other on the reflected line will be perpendicular to the reflection line (these lines are the diagonals as well therefore the diagonals are perpendicular)**

3. Trapezoid: a quadrilateral with at least one pair of parallel sides.
   a. We talked about two different definitions of a trapezoid (we referred to them as "modern" and "traditional". Discuss the differences and explain why the "modern" definition is becoming a preferred way to define a trapezoid.
   b. What is a trapezium? If you are not sure, find its definition on the Internet.
      • Look up the definition of a trapezoid and trapezium in British English. What did you find?
   c. Q1: Certain pairs of adjacent angles have a special property. What is the property and which angles does it apply to? Label the angles to be able to refer to them in your answer.
   d. Q2: Diagonals split a trapezoid into 4 non-overlapping triangles. Among those 4 triangles, there is one pair of similar triangles. Identify the two and explain why they are similar. The other two triangles also share a certain property. What is it and why does it hold? Label the triangles to be able to refer to them in your answer.
   e. Justify the formula for the area of a trapezoid.
   d. Explain how can the following picture be used to derive the formula of a trapezoid.
      • Explain why it is possible to arrange two copies of a trapezoid like this.
      • Explain how this arrangement helps derive the area formula.

4. Parallelogram: a quadrilateral with two pairs of parallel

3. **Answer:**
a. **Traditional:** *A quadrilateral with* **one and only one** *pair of parallel sides.*
   Modern : A quadrilateral with at least one pair of parallel side.
   i. The modern definition is becoming the more prefered way to define a trapezoid, because it's more inclusive of other shapes. The modern definition would allow parallelograms, rectangles, rhombi, and squares to be categorized as a trapezoid.
b. **A quadrilateral with no parallel sides**

|	British	USA
Trapezoid	A quadrilateral with no sides parallel	A quadrilateral with one pair of parallel sides
Trapezium	A quadrilateral with one pair of parallel sides	A quadrilateral with no sides parallel

   i. They're opposite. Source: https://www.mathopenref.com/trapezium.html
   C.

4. Parallelogram: a quadrilateral with two pairs of parallel sides.
    a. Q: Are the diagonals congruent?
    b. Q2: Are the diagonals perpendicular to each other?
    c. Q3: Are the diagonals bisecting each other?
    d. Q4: Is there another property of diagonals worth mentioning?
    e. Justify the formula for the area of a parallelogram.

  4. Answer:
      a. Sometimes
      b. Sometimes
      c. Always
      d. ?
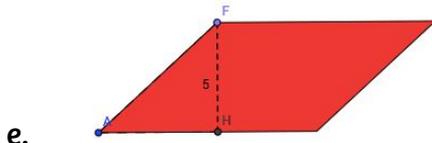
      e. If one moved triangle AFH to the right side of the trapezoid, it would create a rectangle, and a rectangle's area is (base)(height)=area. Therefore, a parallelogram's area is also bh=area

5. Rhombus: a quadrilateral with four congruent sides. (*Use triangle congruence where applicable.*)
    a. Q: Are the diagonals congruent?
    b. Q2: Are the diagonals perpendicular to each other?
    c. Q3: Are the diagonals bisecting each other?
    d. Q4: Diagonals split the rhombus into 4 triangles. They all share certain properties. Name and justify at least one such property.
    e. Prove that a rhombus is a parallelogram. Use only triangle congruence theorems, Isosceles Triangle Theorem and theorems about angles created by two parallel lines cut by a transversal.
6. Rectangle: a quadrilateral with three right angles.
    a. Q: Specify two properties of the diagonals.
7. Square: a quadrilateral with four congruent sides and one right angle.
    a. Q: What might be an alternate formula for the area of a square that does not involve the side(s) of the square? (*See if square is a kite.*)

  5. Answer: Use chart
 6. Answer: Use chart
  7. Answer: Use chart

8. Hierarchy of quadrilaterals: Draw a tree diagram capturing the relationships among quadrilaterals.
9. Name two positives of the modern definition of a trapezoid. (Hint: Think of the (1) hierarchy and (2) trapezoid construction)

8. Answer: See next page
9. Answer: the difference between the "traditional" and "modern" definition is in "exactly" vs. "at least" one pair of parallel sides. The reasons for the recent inclination toward the modern one is that trapezoid properties are inherited down to, say, parallelograms so it makes sense to include parallelograms among trapezoids.

   Another reason is construction - if you construct a trapezoid using its minimal definition, you will see that you can manipulate it to create a parallelogram, rectangle, square. If you want to exclude these shapes from trapezoids, your construction will have to be much more complicated (if possible at all).
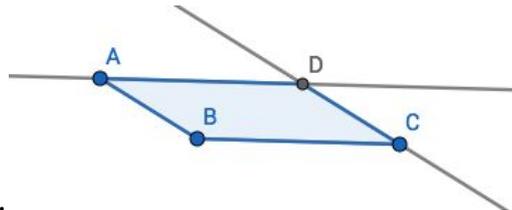
So parallelogram is a trapezoid under the modern definition
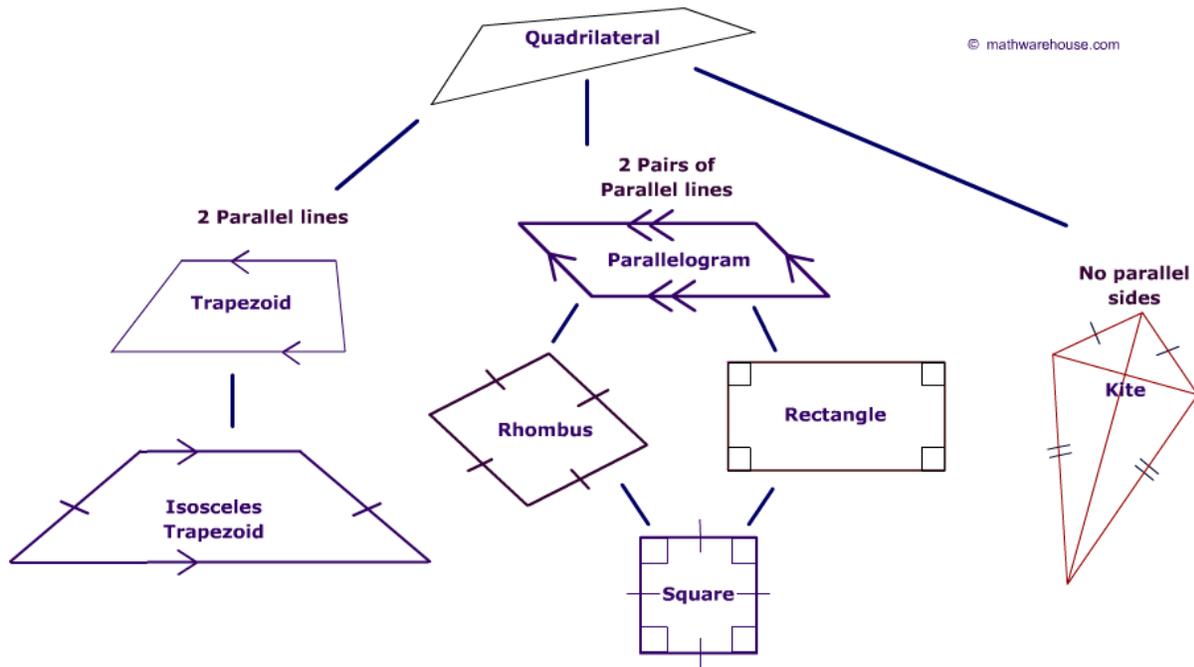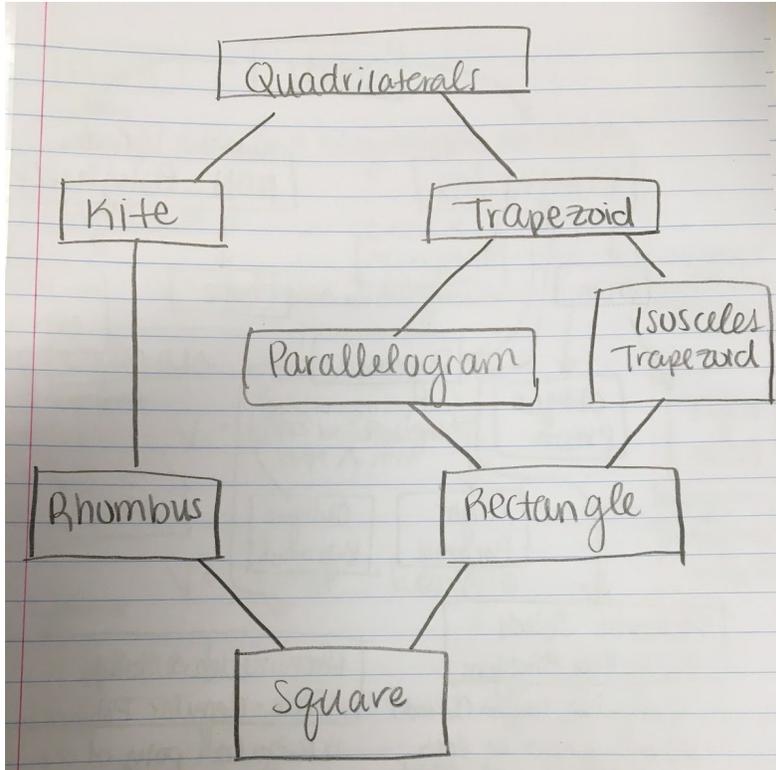
Quadrilateral Constructions

Parallelogram- Create 3 points and segment them together to look like an angle just under 90 degrees. Parallel line the top point. Parallel line the left side and place it on top of point C. Use intersection tool


and then polygon tool.

Square- Create a segment. Compass the segment and put it on both points. Make a perpendicular line at both points. Create intersection point where perpendicular lines and circles meet. Use polygon tool
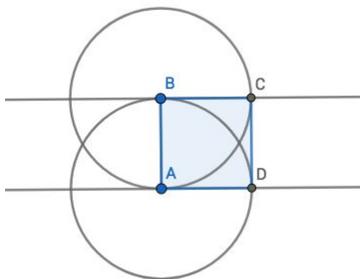
Rectangle- Create a segment. Make a perpendicular line from point A. Put a point on the perpendicular line. Make a perpendicular line from that point. Connect B to D. Use polygon tool.
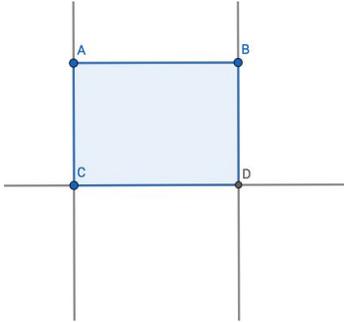
Isosceles trapezoid- Create a segment. Make a parallel line from that segment. Create a perpendicular bisector on original line. Put a point on the parallel line Reflect the new point over the line. Use line
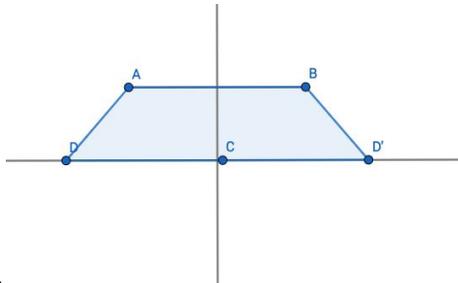
segments to connect.

Trapezoid- Create a segment, create a parallel line. Put two points on the parallel line and connect them to the original segment. Use the polygon tool.
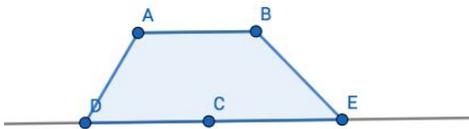
Kite (this way makes it perfectly constrained)- Create two segments that are attached together and look like an obtuse angle. Connect point A to point C. Then use the reflection tool to reflect point B over the line. Use the polygon tool.
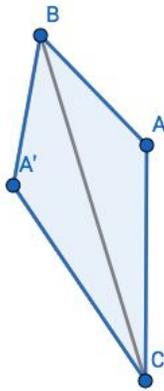
Quadrilateral- Create 4 points. Connect them using the polygon tool.
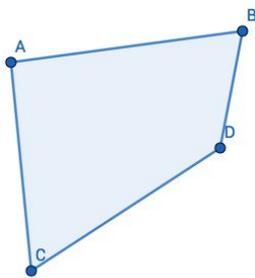
Rhombus- Create a circle with center through point. Create another point on the circle. Use a compass from the point to the center and put it on the point. Do this for both points. Make an intersection point where they intersect inside of the original circle. Connect all 4 points with polygon tool.
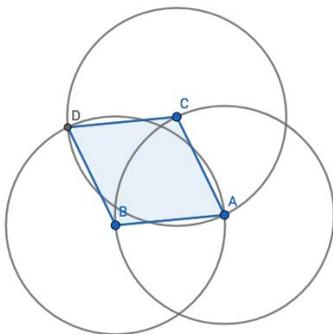
Brianne's applet for quadrilaterals: https://www.geogebra.org/m/bZrbESpE

Area of Kite (using diagonals)

Area

$$\frac{d_1 d_a + d_1 d_b}{2} \quad \frac{d_1 d_b}{2} \Rightarrow \frac{d_1 (d_a + d_b)}{2}$$

$$d_a + d_b = d_2 \therefore \frac{d_1 d_2}{2}$$

	SIDES		ANGLES		DIAGONALS				AREA
	Parallel?	Congruent?	Congruent?	Relationship?	Congruent?	Bisecting?	Perpendicular?	Triangle Partition?	
Kite	Sometimes □-yes ◇-no	2 pairs distinct	1 distinct pair atleast	1 opposite pair	1 opposite pair	atleast 1	yes	yes	$b \cdot h$
Trapezoid	yes	sometimes	sometimes	yes! supplement	sometimes	sometimes	sometimes	atleast 1 pair of similar △'s	$\frac{(b_1+b_2) \cdot h}{2}$
Isosceles Trapezoid	atleast 1 pair	always	always exactly 2 pair.	2 distinct pairs	always	sometimes	sometimes	yes	$\frac{1}{2}(b+a)h$
Parallelogram	2 sets // sides	2 sets cong. sides ASA	2 sets opp. cong. ∠s	adjacent ∠s congruent	sometimes	AAS yes	sometimes	always 1 pair of cong △'s SSS	$b \cdot h$
Rhombus all sides same	always	always	always	2 distinct pairs	sometimes	yes	yes	yes	$\frac{d_1 \cdot d_2}{2}$
Rectangle	yes 2 pairs	yes 2 pair	4 cong ∠s by def.	any 2 angles are supp.	Yes SAS	yes AAS	sometimes	2 pair ≅ △s	$b \cdot h$
Square									