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Developing the Area of a Trapezoid

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A mathematics classroom that reflects the vision of NCTM's *Principles and Standards for School Mathematics* (2001) will have the teacher posing problems, asking questions that build on students' thinking, and encouraging students to explore different solutions. In teaching about area, it is not sufficient to give the students the pertinent formulas and have them merely compute the areas of various polygons. The students should develop an understanding of the concept of area so they can reason about the relationships between shapes to determine area.

According to the Common Core Standards for Mathematics (CCSM), by sixth grade, students are finding the area of special quadrilaterals by composing them into rectangles or decomposing into triangles and other polygons. The Standards for Mathematics Practice from CCSM also describe how students should construct viable arguments and critique the reasoning of others.

One worthwhile task supporting these standards is to ask students to figure out how to find the area of a trapezoid and to justify their method. We have used this ungraded exploration successfully with students from both middle school and high school. Although students do not often get a chance to discover the formula for the area of a trapezoid in their mathematics classrooms, we know that when given an opportunity, students can come up with multiple ways of justifying/discovering this formula (LaSaracina, White, 1999; Wanko, 2005; and Young, 2010).

The activity can be used when the students are being introduced to the areas of special quadrilaterals, especially trapezoids. Prior to engaging with this task, we suggest students explore and develop the formulas for the areas of triangles, rectangles, and parallelograms and have experience decomposing and recomposing polygons. Alternatively, the activity may be approached as forging the connections between the various geometric and algebraic representations, making it a more advanced task, suitable for high school.

The class may be divided into small groups of 2-3 students, provided with poster paper and markers, and challenged to produce at least one way of finding the area of a trapezoid and justifying their method. The teacher's main role is to observe each group, interjecting facilitating questions as appropriate. These questions could include:

1. In what ways could you decompose and recompose this trapezoid in order to create a shape or combination of shapes (such as a triangle, a rectangle, or a parallelogram) for which you already know the area/s?
2. Knowing the area formulas for a rectangle, a parallelogram, and a triangle, what are some ways you could describe the area of this trapezoid? (The question could be posed after students propose a viable plan/sketch for developing the formula.)
3. Is there another way you could decompose and recompose this trapezoid in order to create a shape or combination of shapes (such as a triangle, a rectangle, or a parallelogram) for which you already know the area/s?
4. How is your method similar/different from other methods of deriving the formula for the area of the trapezoid presented by your classmates?

After the exploration, the groups can be asked to display their work on the poster paper attached to the wall and justify their ways of finding the area of a trapezoid. After the methods have been thoroughly contrasted and compared, the class can show that the different formulas generated are equivalent.

One result of this activity is that each student develops an approach to justifying the formula for the area of a trapezoid that they understand and can use to generate the formula, if necessary. Students may prefer one method over another depending on their own level of geometric development and algebraic skills. The role of the teacher in this activity is as facilitator. The teacher may ask follow-up questions to clarify and extend the student's approach. The emphasis is on reasoning and sense making.

The Definition of a Trapezoid

One problem encountered is the differing definitions of a trapezoid. There are two commonly used definitions in the United States. One definition is "A trapezoid is a quadrilateral with *at least* one pair of parallel sides." (Figure 1).

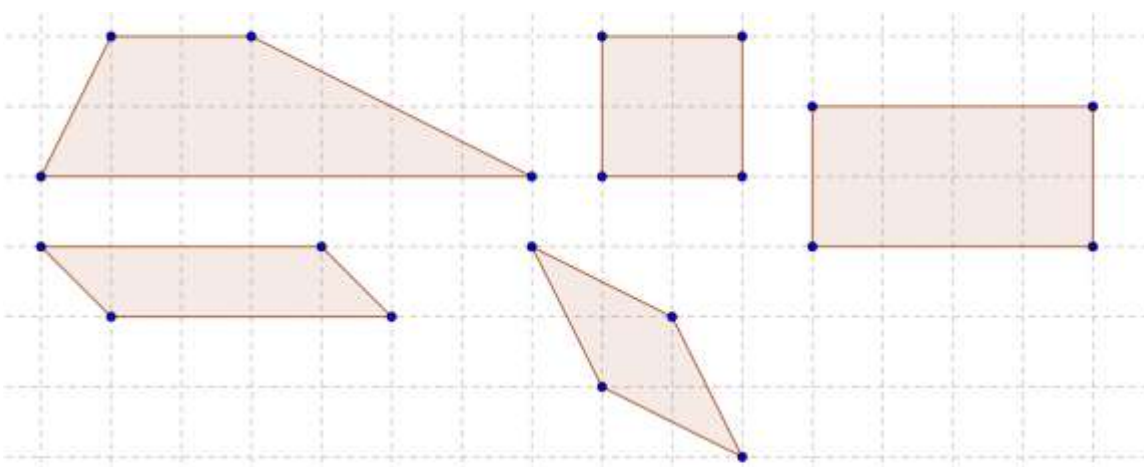

Fig. 1. Definition 1

This definition would classify squares, rhombi, parallelograms, and rectangles as types of trapezoids. The other definition is “A trapezoid is a quadrilateral with *exactly* one pair of parallel sides.” This definition excludes squares, rhombi, parallelograms, and rectangles from being classified as types of trapezoids. (Figure 2).

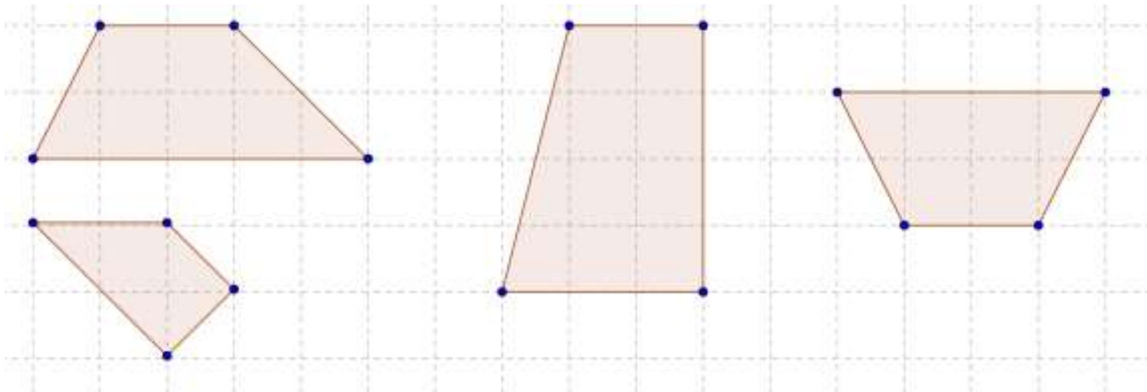

Fig. 2. Definition 2

The formula for the area of a trapezoid $A = \frac{1}{2}(b_1 + b_2)h$ where A is the area of the trapezoid, b_1 and b_2 are the bases, and h is the height is appropriate for both definitions of trapezoid. One can actually use this formula to find the area of a square, a rhombus, a parallelogram, or a rectangle. Although the definition mandated by our state Department of Education and, subsequently, our activity, is “A trapezoid is a quadrilateral with *exactly* one pair of parallel sides”, either definition is usable with our students’ examples.

Special Cases

When presented with the task of developing a formula to calculate the area of a trapezoid, our students initially developed solutions that worked only for special cases of trapezoids, such as isosceles trapezoids and/or trapezoids with right angles.

An isosceles trapezoid has two congruent, non-parallel sides. Because of the definition of trapezoid that we are using, the bases cannot be congruent or the figure would be a parallelogram and not considered a trapezoid. One of the common solutions presented was to cut off two right triangles, leaving the rectangle (Figure 3). Using a different approach, students cut a right triangle from one end of the trapezoid and moved it to the opposite end to create a rectangle. This method worked only for isosceles trapezoids (Figure 4).

Fig. 3. Student solution 1

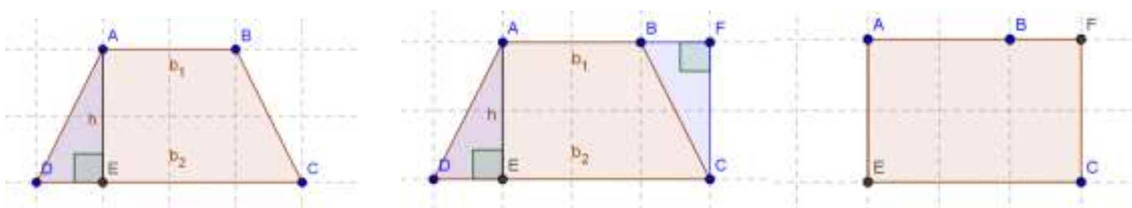

Fig 4. Student solution 2

Fig. 5. Student solution 3

As shown in Figure 5, students also began with a trapezoid containing a right angle and showed that it can be decomposed into a rectangle and a right triangle. Then they added the area of the rectangle and triangle to derive the formula for the area of a trapezoid. This strategy only works for the special case of trapezoids with right angles.

As students presented aforementioned strategies that only addressed the special cases, we followed up with questions that encouraged all students to critically evaluate their strategies for a general case of a trapezoid. This created an opportunity for all students to engage in mathematics reasoning at a deeper level and allowed us to facilitate a meaningful and rich mathematical discourse.

The General Case

Next, we challenged our students to develop a formula to compute the area of **any** trapezoid. When presented with this task, the students responded as seen in the examples below. A teacher should not expect to see as many approaches to developing the formula for finding the area of a trapezoid within a single class as are seen in Figures 6, 7, and 8. It is also unlikely that the students will all produce the same solution. In the remote case that they do only create one solution, the teacher can provide examples of other solutions and challenge the students to justify the appropriateness of these solutions. This may prompt the students to develop additional solutions beyond what they have been shown.

The student strategies separated into three types: Type 1, Type 2, and Type 3. Type 1 includes proofs in which students decomposed a trapezoid into one or more of the following shapes: triangles, rectangles, parallelograms, and found the area of a trapezoid as a sum of the areas of these shapes.

Type 1

Below you will see the examples of student strategies that fall under the Type 1 category and thumbnail sketches of the reasoning provided by the students. (Figure 6):

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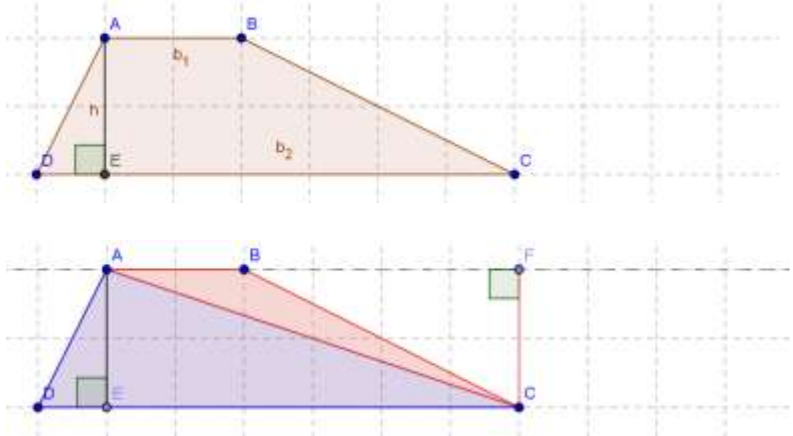

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Students decomposed the trapezoid into two triangles ACB and ACD. The area of $\triangle ACB = \frac{1}{2}b_1h$. The area of $\triangle ACD = \frac{1}{2}b_2h$. Combining the areas of these two triangles to make the trapezoid $ABCD$, the area is $\frac{1}{2}b_2h + \frac{1}{2}b_1h$ = $\frac{1}{2}(b_1 + b_2)h$.
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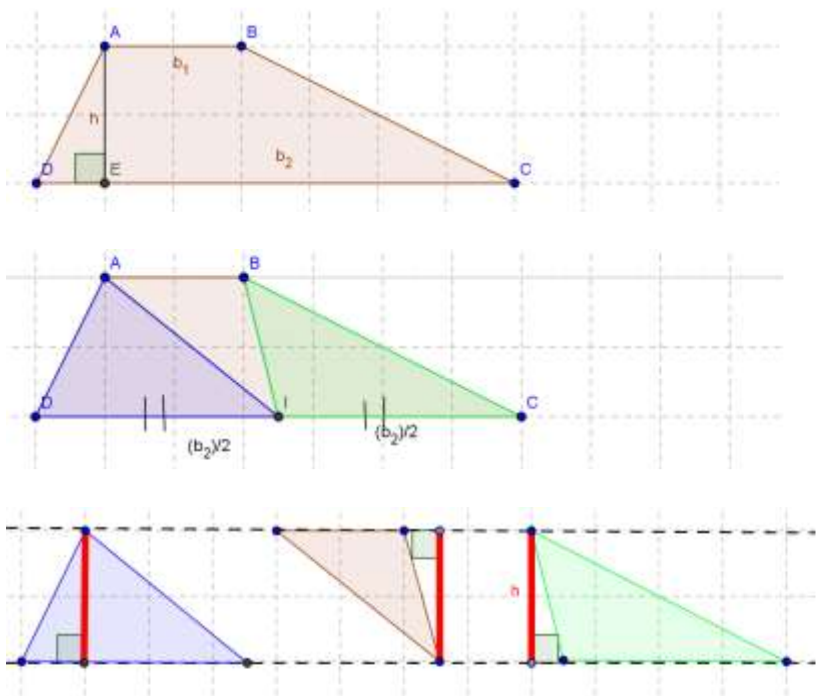

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In this method, the students connected the midpoint I of the longer base to the vertices of the other base, decomposing the trapezoid into three triangles, ADI, AIB, and BIC. Since I is the midpoint of the base b_2, $\overline{DI}$ and $\overline{IC}$ both equal $\frac{b_2}{2}$. So, the area of $\triangle ADI = \frac{1}{2}\left(\frac{b_2}{2}\right)h$. The area of $\triangle AIB = \frac{1}{2}b_1h$. The area of $\triangle BIC = \frac{1}{2}\left(\frac{b_2}{2}\right)h$.
Since the area of trapezoid $ABCD$ may be found by combining the areas of the three triangles, it equals $\frac{1}{2}\left(\frac{b_2}{2}\right)h + \frac{1}{2}b_1h + \frac{1}{2}\left(\frac{b_2}{2}\right)h$.
This expression can be simplified to $\frac{1}{2}(b_1 + b_2)h$
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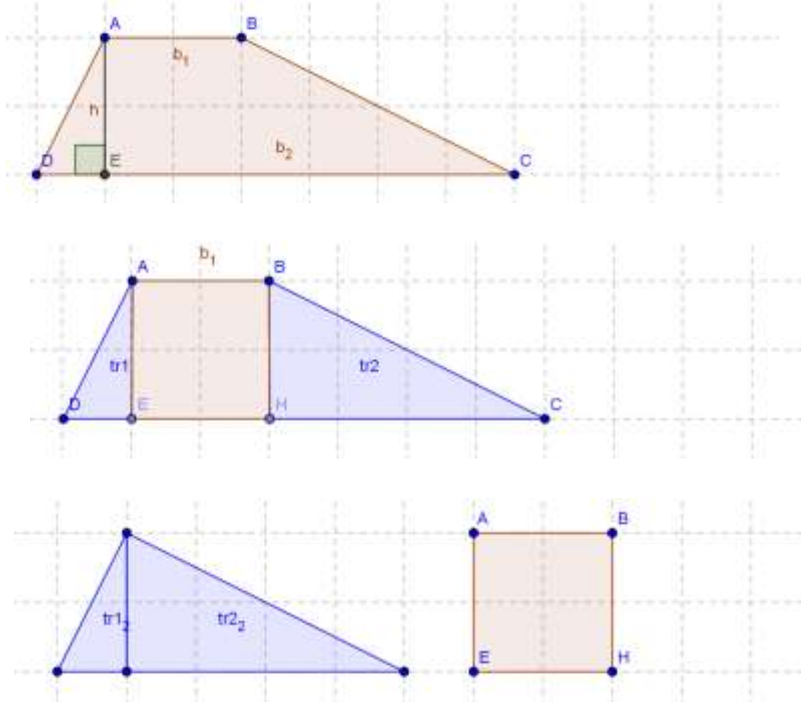

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<td>
The students dropped perpendiculars from the endpoints of the shorter base $\overline{AB}$ to the opposite base, forming the rectangle $ABHE$ and the right triangles AED and BHC. The base of rectangle $ABHE$ is b_1, and its height is h. So, its area is b_1h. The two triangles can be slid together so that $\overline{AE}$ from triangle AED and $\overline{BH}$ from triangle BHC coincide, forming the altitude of a new triangle. The base of the new triangle is $b_2 - b_1$ since the base of the rectangle b_1 has been removed from original base b_2 of the trapezoid. The area of the new triangle is $\left(\frac{1}{2}(b_2 - b_1)\right)h$. The area of the original trapezoid is the sum of the areas of the rectangle and the triangle: b_1h + $\left(\frac{1}{2}(b_2 - b_1)\right)h$ or $\left(b_1 + \frac{1}{2}b_2 - \frac{1}{2}b_1\right)h$ or $\frac{1}{2}(b_1 + b_2)h$.
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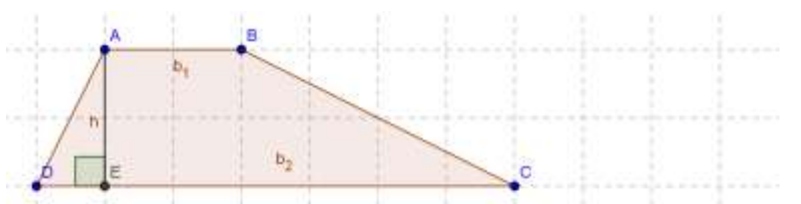

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The students constructed a line through point B on base b_1 parallel to leg $\overline{AD}$, decomposing the trapezoid $ABCD$ into parallelogram $ABFD$ and $\triangle BCF$. The area of parallelogram $ABFD$ is b_1h. The area of $\triangle BCF$ is $\frac{1}{2}(b_2 - b_1)h$. The sum of the areas of parallelogram $ABFD$ and $\triangle BCF$ is
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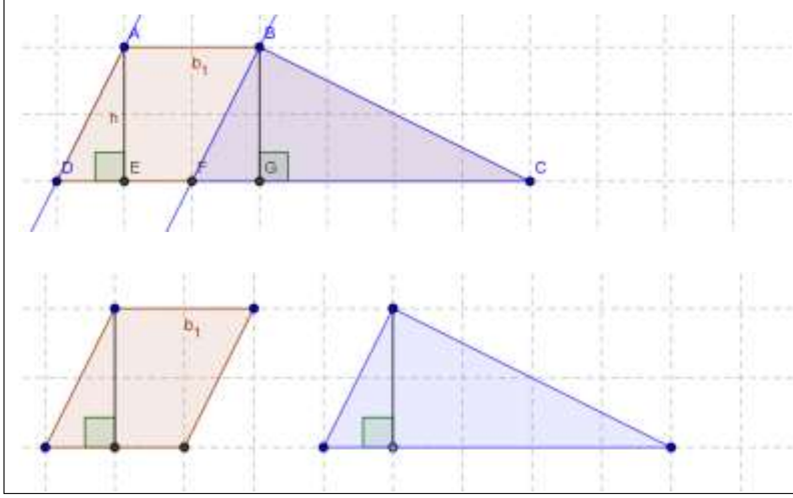
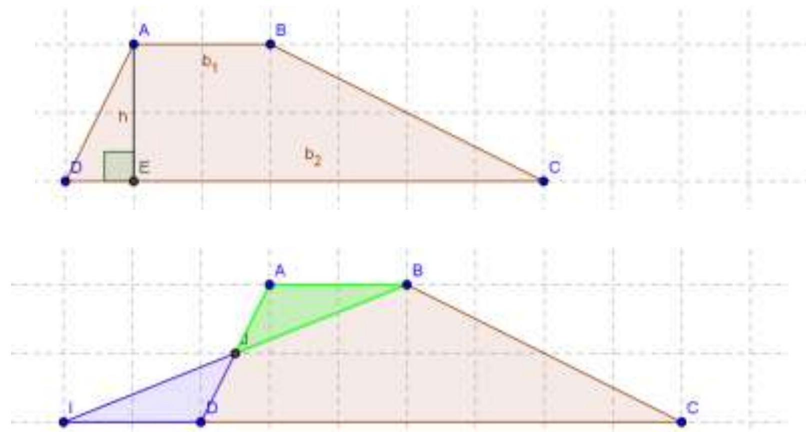
	$b_1h + \frac{1}{2}\left(b_2 - b_1\right)h$ or $\frac{1}{2}(b_1 + b_2)h$

Fig. 6. Type 1 strategies

Type 2

In the second type of strategy students derived the formula for the area of the trapezoid by using transformational geometry. Below are four examples of Type 2 strategies developed by our students and summaries of their reasoning (Figure 7).

	In this example, the students connected vertex B in trapezoid $ABCD$ to the midpoint J of the side $\overline{AD}$. They then rotated $\triangle ABJ$ 180° about J, forming $\triangle BCI$. The area of $\triangle BCI = \frac{1}{2}(\overline{ID} + \overline{DC})h$ $= \frac{1}{2}(b_1 + b_2)h$.

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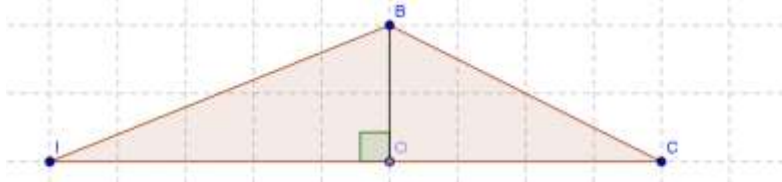
<td>
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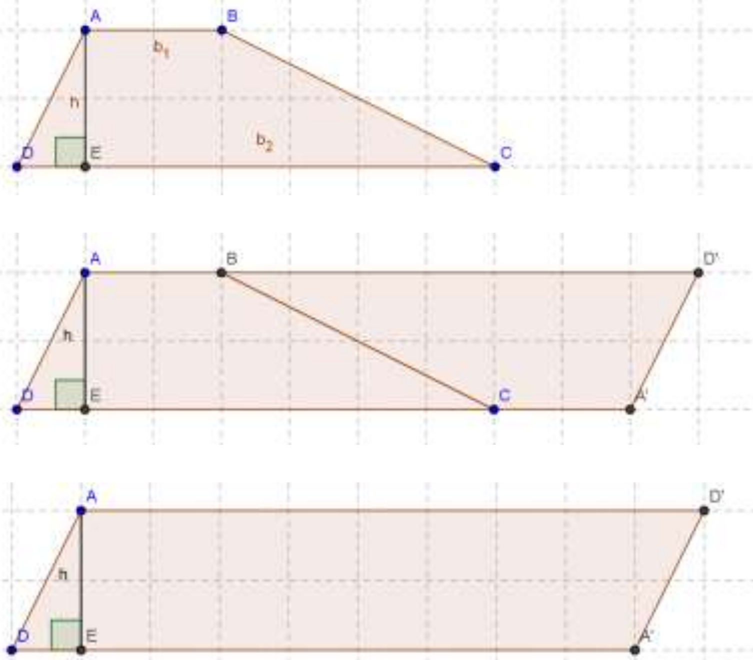
<td>
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<td>In this example, students duplicated trapezoid $ABCD$ and rotated it 180° around the midpoint of side $\overline{BC}$, forming $AD'A'D$. The length of bases AD' and DA' is $b_1 + b_2$. $\overline{AD} \cong \overline{A'D'}$. Since $AD'A'D$ is a quadrilateral with opposite sides congruent, it is a parallelogram. So, the area of parallelogram $AD'A'D$ is $(b_1 + b_2)$h. Since the area of trapezoid $ABCD$ is half the area of parallelogram $AD'A'D$, it is $\frac{1}{2}(b_1 + b_2)h$.</td>
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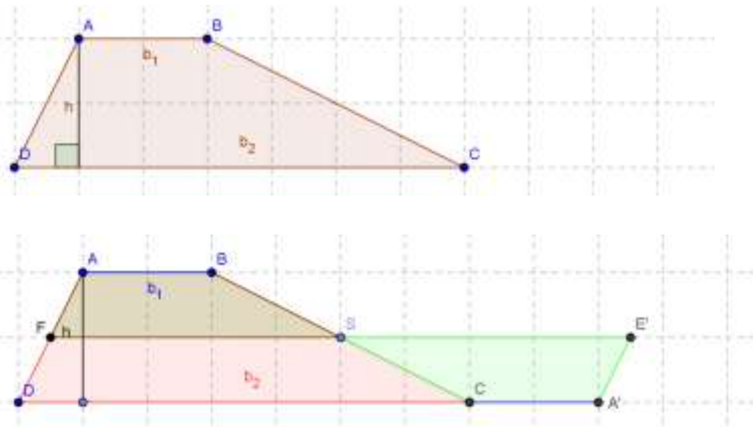
<td>
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<td>In this solution, the students constructed the line segment connecting the midpoints F and S of the legs of trapezoid $ABCD$ forming the median (also known as midsegment or midline). They then rotated trapezoid $ABSF$ 180° around point S to form parallelogram $FE'A'D$, which has the same area as the original trapezoid $ABCD$. The area of parallelogram $FE'A'D$ is</td>
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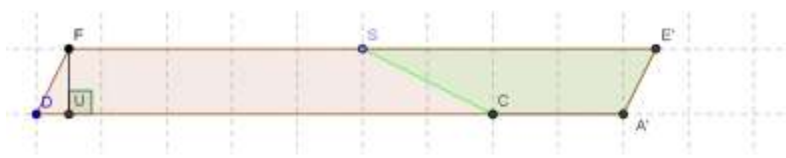
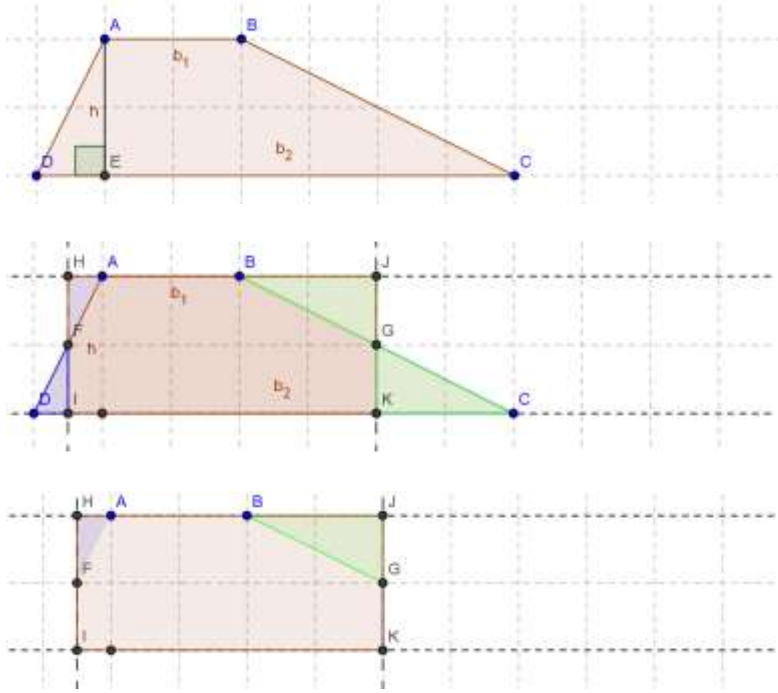
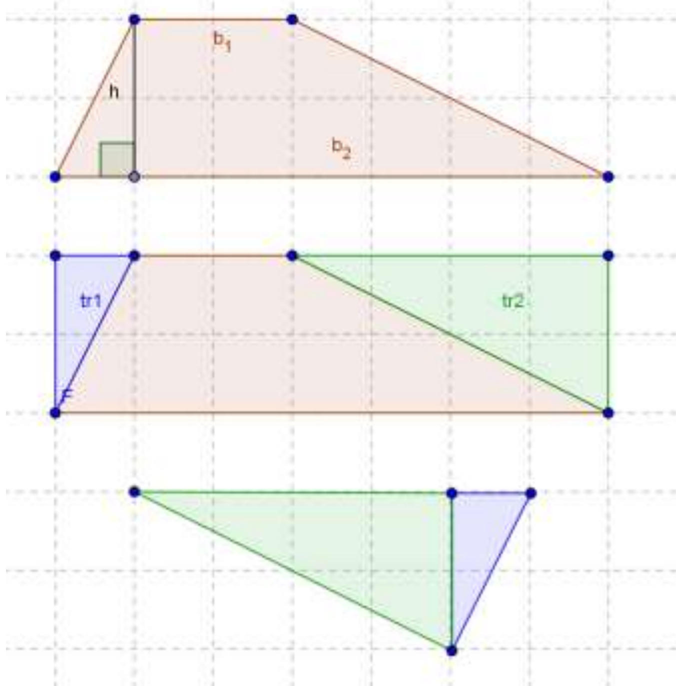
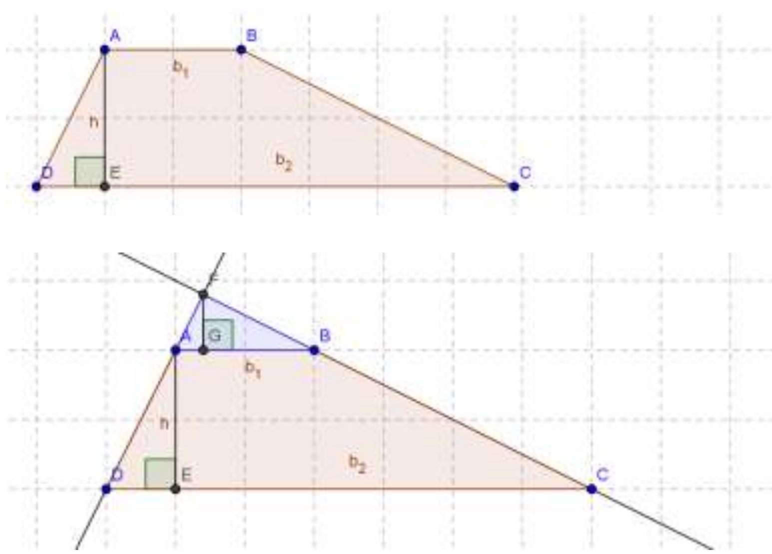
	$(b_2 + b_1)\left(\frac{1}{2}h\right)$. This expression can be rewritten as $\frac{1}{2}(b_1 + b_2)h$.
	The students form a rectangle by dropping a perpendicular line segment from the midpoint of each leg, forming ΔFID and ΔGCK. Each of these triangles is rotated 180° around their vertex that was the midpoint of the leg to form the rectangle $HJKI$, that has the same area as the original trapezoid $ABCD$. The base of the rectangle is the average of the two bases of the trapezoid, $\frac{b_1 + b_2}{2}$. Therefore, the area of the rectangle (and of the trapezoid) is $\left(\frac{b_1 + b_2}{2}\right)h$ or $\frac{1}{2}(b_1 + b_2)h$.

Fig. 7. Type 2 strategies

Type 3

Finally, the third type of student strategy for deriving the formula for the area of a trapezoid includes the action of enclosing the original trapezoid into a constructed larger shape, such as a rectangle, or parallelogram, or a triangle. Then students found the area of a trapezoid by subtracting the area of additional (excess) pieces from a constructed larger shape. We present three examples of solutions involving Type 3 strategies below (Figure 8).

	The students constructed perpendiculars from endpoints of the base b_2 to the extended base b_1, forming a rectangle. The area of the rectangle is b_2h. The area of the triangle formed by combining the two added triangles is $\frac{1}{2}(b_2 - b_1)h$. The area of the trapezoid can be calculated by subtracting the area of the added triangles from the area of the rectangle or $b_2h - \frac{1}{2}(b_2 - b_1)h$ or $\frac{1}{2}(b_1 + b_2)h$
	The students extended leg $\overline{DA}$ and leg $\overline{CB}$ until they met at point F, forming $\triangle FDC$. $\overline{FG}$ is the altitude (h_2) of newly formed $\triangle FAB$. The area of $\triangle FAB$ is $\frac{1}{2}b_1h_2$. The area of $\triangle FDC$ is $\frac{1}{2}b_2(h + h_2)$. The area of the original trapezoid $ABCD$ is the difference between the area of $\triangle FDC$ and $\triangle FAB$ or $\frac{1}{2}b_2(h + h_2) - \frac{1}{2}b_1h_2$.

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<tr>
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<td>Simplifying, the area is $\frac{1}{2}b_2h + \frac{1}{2}b_2h_2 - \frac{1}{2}b_1h_2 =$ $\frac{1}{2}b_2h + \frac{1}{2}\left(b_2 - b_1\right)h_2$.
The line segment $\overline{AB}$ is parallel to the base of $\triangle FCD$. So, $\triangle FAB$ is similar to $\triangle FDC$.
Therefore, $\frac{h_2}{b_1} = \frac{h_2 + h}{b_2}$.
Solving for h_2, $h_2 =$ $\frac{b_1h}{b_2 - b_1}$. Substituting this value for h_2 into the expression for area of the trapezoid, $\frac{1}{2}b_2h +$ $\frac{1}{2}\left(b_2 - b_1\right)\left(\frac{b_1h}{b_2 - b_1}\right)$. This expression can be simplified to $\frac{1}{2}b_2h + \frac{1}{2}b_1h$ or $\frac{1}{2}(b_1 + b_2)h$.</td>
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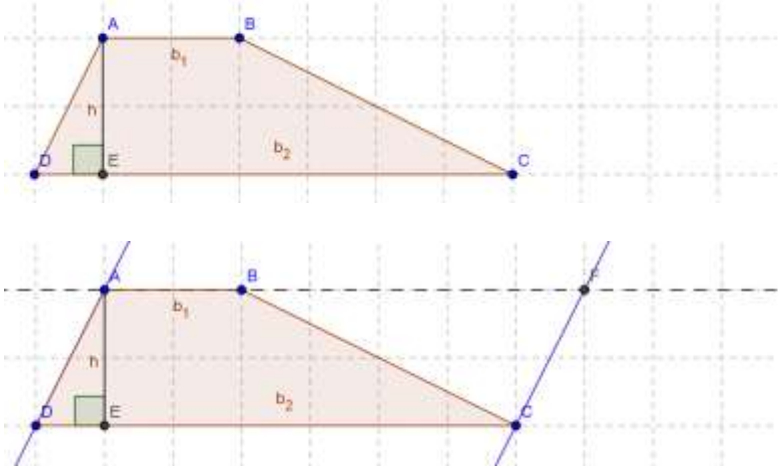
<td>
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<td>In this example, students constructed a line through point C parallel to leg $\overline{AD}$. They extended base b_1 to intersect this line at point F, forming parallelogram $AFCD$.
The area of this parallelogram is b_2h.
The area of $\triangle BFC$ is $\frac{1}{2}\left(b_2 - b_1\right)h$. The area of trapezoid $ABCD$ is the difference of the area of parallelogram $AFCD$ and $\triangle BFC$ or b_2h -</td>
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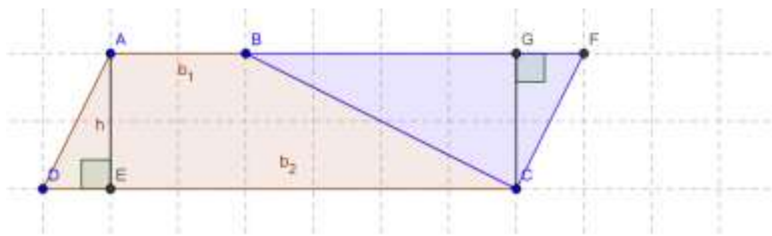	$\frac{1}{2}(b_2 - b_1)h$. This expression can be simplified to $b_2h - \frac{1}{2}b_2h + \frac{1}{2}b_1h$ or $\frac{1}{2}(b_1 + b_2)h$

Fig. 8. Type 3 strategies

This activity is flexible and can be used with students who have various mathematical backgrounds. Depending on student's level of geometric development and her/his algebraic skills, she/he may prefer one type of strategy over another. Asking students to think about strategies presented by peers and compare those to their own can generate a rich mathematical discussion. It can provide them with an opportunity to reason mathematically and communicate their ideas.

Conclusion

This activity can be viewed in the context of reasoning and sense making as described in the Standards for Mathematical Practice from CCSM. In this exploration, students have a chance to consider special cases of trapezoids that simplify the given tasks, allowing them to gain insight into more generic examples. The approach an individual student uses when trying to determine how to find the area of a trapezoid may depend on his/her van Hiele Level of Geometric Thinking. Comparing the van Hiele Levels of Geometric Thinking (Mason, 1998) to the types of solutions displayed, we surmise that students who are thinking at the Visualization Level (Level 1), where shapes are recognized by appearance alone, may decompose the trapezoid into shapes that they are familiar with such as triangles and rectangles. Students that are thinking at the Analysis Level (Level 2), where properties of shapes are perceived but are isolated and unrelated, may try to use the fact that the altitude of the trapezoid is perpendicular to its bases or that the bases of the trapezoid are parallel in their approach to solving the problem. Students who are thinking at the Abstraction Level (Level 3), where definitions are meaningful with relationships perceived between figures and between properties, are more capable of creating approaches that involve transformations and addition of figures (Type 2 and Type 3). The variations in approach are to be expected and even encouraged. What is most important is that students are able to justify their thinking.

Mathematically proficient students can determine the correspondence between geometric manipulations and algebraic equations. Students not as proficient might prefer to physically manipulate objects in order to see the connections between the area of the trapezoid and its algebraic formula. Mathematically proficient students can develop more than one approach to producing the area of a trapezoid and will be able to see correspondences between different methods.

The problem of finding a method to calculate the area of any trapezoid involves a wide range of important ideas in geometry and measurement such as composition and decomposition, transformations, polygons, perpendicular and parallel lines, and similarity and congruency. The activity provides the opportunity to explore these ideas and their relationships and to integrate algebra and geometry. It is a worthwhile task that can develop the foundation for further geometric understanding, modeling, and proof, especially reasoning and sense making. It requires active participation of all students and allows teachers to act as facilitators who can create a rich mathematical discourse in the classroom. The model of this activity could also be used to design other activities that promote reasoning and sense-making.

References

Common Core State Standards Initiative: Mathematics, http://www.corestandards.org/the-standards/mathematics.

LaSaracina, Barbara A. and Sharon K. White. "The Restless Triangle and the Transforming Trapezoid." *Teaching Children Mathematics* 5, no. 6 (1999): 336-7, 366.

Mason, Marguerite M. "The van Hiele Levels of Geometric Understanding." In *The Professional Handbook for Teachers: Geometry*, 4-8, Boston: McDougal-Littell/Houghton-Mifflin, 1998.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000.

Wanko, Jeffrey J. "Tapping into Trapezoids." *Mathematics Teacher* 99, no. 3 (2005): 190-5.

Young, Elaine. "Trapezoids to Triangles." *Mathematics Teaching in the Middle School* 15, no. 7 (2010): 414-7.