CHAPTER **NINE**

Circle Geometry

Circles have already been studied using coordinate methods, and circles were essential in the development of the trigonometric functions. Many important properties of circles, however, remain to be developed, and the methods of Euclidean geometry are particularly suited to this task — first, the circle is easily defined geometrically in terms of centre and radius, compasses being designed to implement this definition, and secondly, angles are handled far more easily in Euclidean geometry than in coordinate geometry.

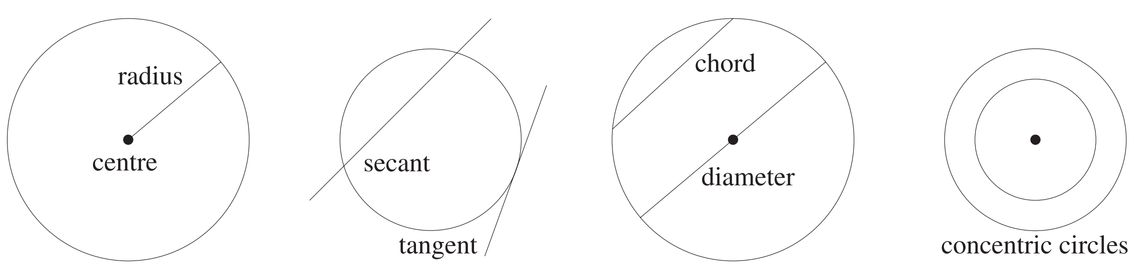
Study Notes: Although this material may be familiar from earlier years, the emphasis now is less on numerical work and more on the logical development of the theory and on its applications to the proof of further results. Most students will therefore find the chapter rather demanding. Sections 9A–9D deal with angles at the centre and circumference of circles. Three difficult converse theorems here are quite new — these converses concern the circumcircle of a right triangle, and two tests for the concyclicity of four points. Sections 9E–9G then examine tangents to circles and the angles they form with diameters and chords.

As in the previous chapter, all the course theorems have been boxed. Some proofs are written out in the notes, and some are presented in structured questions placed at the start of the following development section. All these proofs are important — working through these proofs is an essential part of the course.

Some of the Extension sections of these exercises are longer than normal, but 3 Unit students should be reassured that these questions, as always, are beyond the standards of the 3 Unit HSC papers. The 4 Unit HSC papers usually contain a difficult geometry question, and many of the standard results associated with these questions have therefore been included in the Extension sections.

9 A Circles, Chords and Arcs

The first group of theorems concern angles at the centre of a circle and their relationship with chords and arcs. The section ends with the crucial theorem that any set of three non-collinear points lie on a unique circle. First, some definitions:



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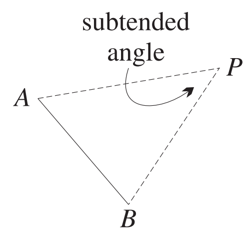
1 CIRCLE, CENTRE, RADIUS, TANGENT, SECANT, CHORD, DIAMETER:
- A *circle* is the set of all points that are a fixed distance (called the *radius*) from a given point (called the *centre*).
- A *radius* is the interval joining the centre and any point on the circle.
- A *tangent* is a line touching a circle in one point.
- A *secant* is the line through two distinct points on a circle.
- A *chord* is the interval joining two distinct points on a circle.
- A *diameter* is a chord through the centre.
- Two circles with a common centre are called *concentric*.

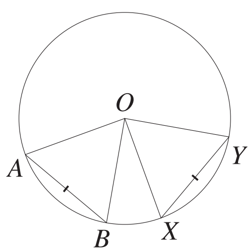
Subtended angles: We shall speak of *subtended angles* throughout this chapter, particularly angles subtended by chords of circles at the centre and at a point on the circumference.



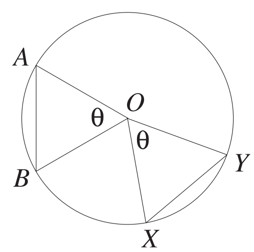
2 ANGLES SUBTENDED BY AN INTERVAL: The *angle subtended at a point P by an interval AB* is the angle $\angle APB$ formed at P by joining AP and BP.

A Chord and the Angle Subtended at the Centre: The straightforward congruence proofs of this theorem and its converse have been left to the following exercise.

3 COURSE THEOREM: In the same circle or in circles of equal radius:
- Chords of equal length subtend equal angles at the centre.
- Conversely, chords subtending equal angles at the centre have equal lengths.



$\angle AOB = \angle XOY$
(equal chords AB and XY subtend equal angles at the centre O).

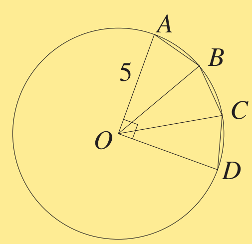


$AB = XY$
(chords subtending equal angles at the centre O are equal).

WORKED EXERCISE: In the diagram below, the chords AB, BC and CD have equal lengths. Prove that $AC = BD = 5$, then find AD.

SOLUTION: The three equal chords subtend equal angles at the centre O,
so $\angle AOB = \angle BOC = \angle COD = 30°$,
and $\angle AOC = 60°$.
But $OA = OC$ (radii),
so $\triangle OAC$ is equilateral, and $AC = 5$.
Similarly, $\triangle OBD$ is equilateral, and $BD = 5$.

Secondly, $AD^2 = 5^2 + 5^2$ (Pythagoras),
hence $AD = 5\sqrt{2}$.



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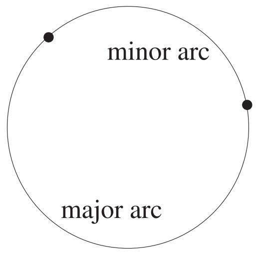
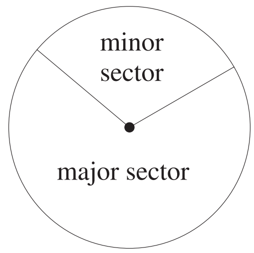
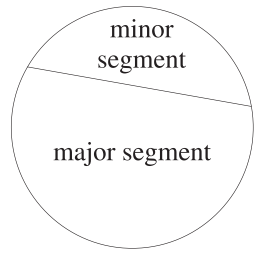
Arcs, Sectors and Segments: Here again are the basic definitions.

4 ARCS, SECTORS AND SEGMENTS:
- Two points on a circle dissect the circle into a *major arc* and a *minor arc*, called *opposite arcs*.
- Two radii of a circle dissect the region inside the circle into a *major sector* and a *minor sector*, called *opposite sectors*.
- A chord of a circle dissects the region inside the circle into a *major segment* and a *minor segment*, called *opposite segments*.



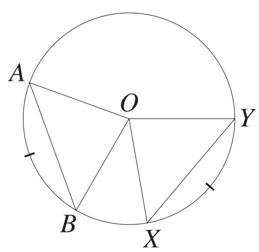




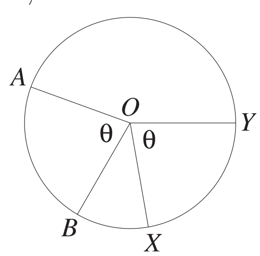
A fundamental assumption of the course is that arc length is proportional to the angle subtended at the centre. In particular, we shall assume that:

5 COURSE ASSUMPTION: In the same circle or in circles of equal radius:
- Equal arcs subtend equal angles at the centre.
- Conversely, arcs subtending equal angles at the centre are equal.
- Equal arcs cut off equal chords.
- Conversely, equal chords cut off equal arcs.

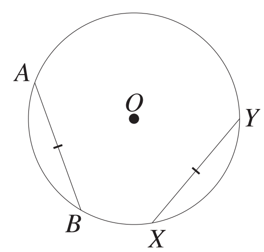
The first two statements can be proven informally by rotating one arc onto the other. The last two statements then follow from the first two, using the previous theorem. In the following diagrams, O is the centre of each circle.



$\angle AOB = \angle XOY$ and $AB = XY$ (arcs AB and XY are equal).



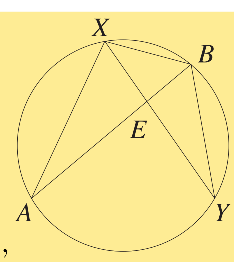
arc AB = arc XY (arcs subtending equal angles at the centre are equal).



arc AB = arc XY (equal chords AB and XY cut off equal arcs).

WORKED EXERCISE: Two equal chords AB and XY of a circle intersect at E. Use equal arcs to prove that $AX = BY$ and that $\triangle EBX$ is isosceles.

SOLUTION:



First, arc AB = arc XY (equal chords cut off equal arcs),
so arc AX = arc BY (subtracting arc XB from each arc),
so $AX = BY$ (equal arcs cut off equal chords).
Secondly, $\triangle ABX \equiv \triangle YXB$ (SSS),
so $\angle ABX \equiv \angle YXB$ (matching angles of congruent triangles),
hence $EX = EB$ (opposite angles are equal).

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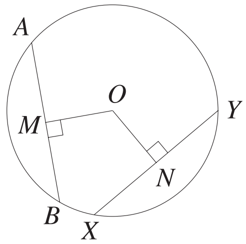
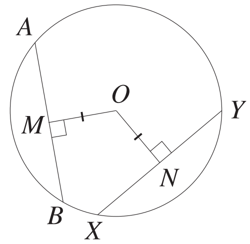
Chords and Distance from the Centre: The following theorem and its converse about the distance from a chord to the centre are often combined with Pythagoras' theorem in mensuration problems about circles. They are proven in the exercises.

6 COURSE THEOREM: In the same circle or in circles of equal radius:
- Equal chords are equidistant from the centre.
- Conversely, chords that are equidistant from the centre are equal.



If $AB = XY$,
then $OM = ON$ (equal chords are equidistant from the centre O).



If $OM = ON$,
then $AB = XY$ (chords equidistant from the centre O are equal).

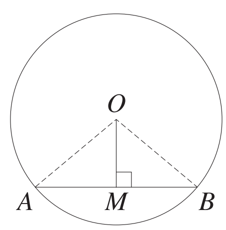
Chords, Perpendiculars and Bisectors: The radii from the endpoints of a chord are equal, and so the chord and the two radii form an isosceles triangle. The following important theorems are really restatements of theorems about isosceles triangles.

7 COURSE THEOREM:
- The perpendicular from the centre of a circle to a chord bisects the chord.
- Conversely, the interval from the centre of a circle to the midpoint of a chord is perpendicular to the chord.
- The perpendicular bisector of a chord of a circle passes through the centre.

PROOF:

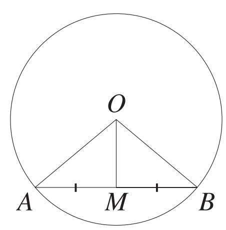
A. To prove the first part, let AB be a chord of a circle with centre O.
Let the perpendicular from O meet AB at M. We must prove that $AM = MB$.
In the triangles AMO and BMO:
1. $OM = OM$ (common),
2. $OA = OB$ (radii),
3. $\angle OMA = \angle OMB = 90^\circ$ (given),

so $\triangle AMO \equiv \triangle BMO$ (RHS).
Hence $AM = BM$ (matching sides of congruent triangles).



B. To prove the second part, let AB be a chord of a circle with centre O.
Let M be the midpoint of AB. We must prove that $OM \perp AB$.
In the triangles AMO and BMO:
1. $OM = OM$ (common),
2. $OA = OB$ (radii),
3. $AM = BM$ (given),

so $\triangle AMO \equiv \triangle BMO$ (SSS).
Hence $\angle AMO = \angle BMO$ (matching angles of congruent triangles).
But AMB is a straight line, and so $\angle AMO = 90^\circ$.



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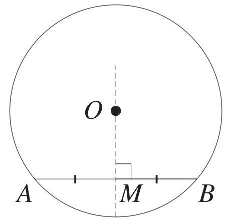
C. To prove the third part, let AB be a chord of a circle with centre O. As proven in the first part, the perpendicular from O to AB bisects AB, and hence is the perpendicular bisector of AB. Hence the perpendicular bisector of AB passes through O, as required.

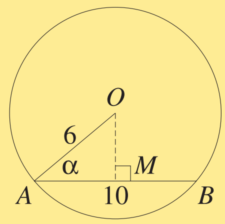


WORKED EXERCISE: In a circle of radius 6 units, a chord of length 10 units is drawn.

(a) How far is the chord from the centre?

(b) What is the sine of the angle between the chord and a radius at an endpoint of the chord?

SOLUTION: Let the centre be O and the chord be AB. Construct the perpendicular OM from O to AB, and join the radius OA.

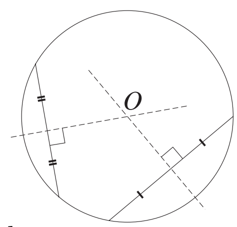


(a) Then $AM = MB$ (perpendicular from centre to chord),
so $OM^2 = 6^2 - 5^2$ (Pythagoras),
and $OM = \sqrt{11}$.

(b) Also, $\sin\alpha = \frac{1}{6}\sqrt{11}$.

Constructing the Centre of a Given Circle: The third part of the previous theorem gives a method of constructing the centre of a given circle.

8 COURSE CONSTRUCTION: Given a circle, construct any two non-parallel chords, and construct their perpendicular bisectors. The point of intersection of these bisectors is the centre of the circle.



PROOF: Since every perpendicular bisector passes through the centre, the centre must lie on every one of them, so the centre must be their single common point.

Constructing the Circle through Three Non-collinear Points: Any two distinct points determine a unique line. Three points may or may not be collinear, but if they are not, then they lie on a unique circle, constructed as described here.

9 COURSE THEOREM: Given any three non-collinear points, there is one and only one circle through the three points. Its centre is the intersection of any two perpendicular bisectors of the intervals joining the points.

The circle is called the *circumcircle* of the triangle formed by the three points, and its centre is called the *circumcentre*.

GIVEN: Let ABC be a triangle, and let O be the intersection of the perpendicular bisectors OP and OQ of BC and CA respectively.

AIM: To prove:
A. The circle with centre O and radius OC passes through A and B.
B. Every circle through A, B and C has centre O and radius OC.

CONSTRUCTION: Join AO, BO and CO.

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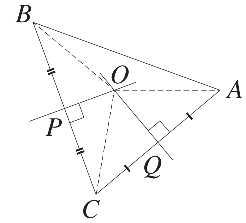
PROOF:

A. In the triangles BOP and COP:



1. $OP = OP$ (common),
2. $BP = CP$ (given),
3. $\angle BPO = \angle CPO = 90^\circ$ (given),

so $\triangle BOP \equiv \triangle COP$ (SAS).

Hence $BO = CO$ (matching sides of congruent triangles).

Similarly, $\triangle AOQ \equiv \triangle COQ$ and $AO = CO$.

Hence $BO = CO = AO$,

and the circle with centre O and radius OC passes through A and B.

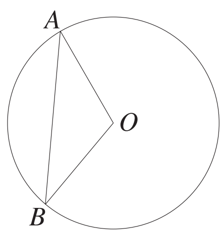
B. Now suppose that some circle with centre Z passes through A, B and C. We have already shown that the perpendicular bisector of a chord passes through the centre, and so Z lies on both OP and OQ. Hence O and Z coincide, and the radius is OC.

Exercise **9A**

NOTE: In each question, all reasons must always be given. Unless otherwise indicated, any point labelled O is the centre of the circle.

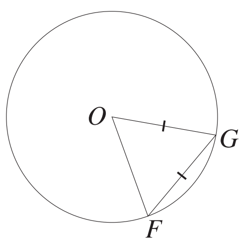
1. In part (c), O and Z are the centres of the two circles of equal radii.

(a)



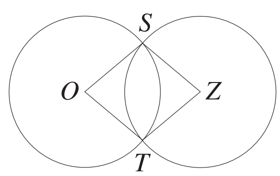
Prove that $\triangle OAB$ is isosceles.

(b)



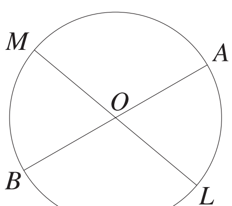
Prove that $\triangle OFG$ is equilateral.

(c)



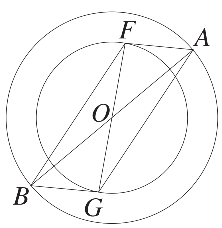
Prove that $OSZT$ is a rhombus.

(d)



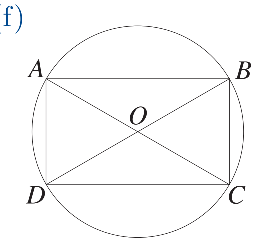
Prove that arcs AL and MB have equal lengths.

(e)



Prove that $AFBG$ is a parallelogram.

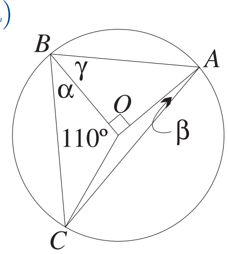
(f)



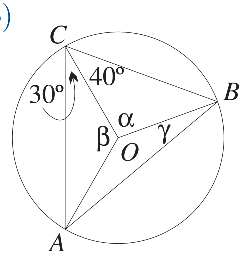
Prove that $ABCD$ is a rectangle.

2. Find α, β, γ and δ. In parts (g) and (h), prove that arc ABC = arc BCD and $AC = BD$.

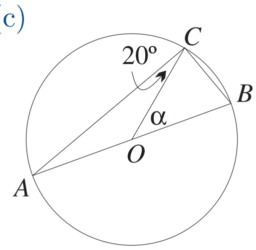
(a)



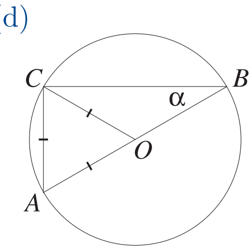
(b)



(c)



(d)



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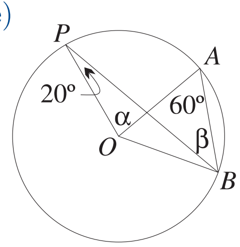
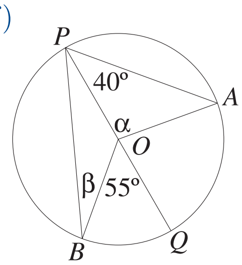
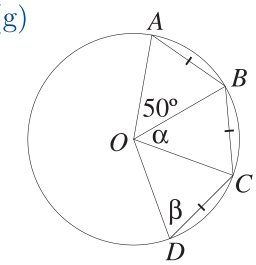
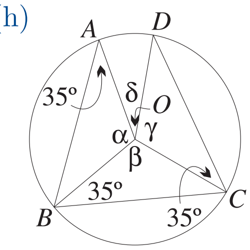
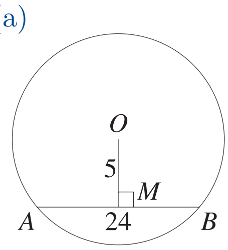
(e)


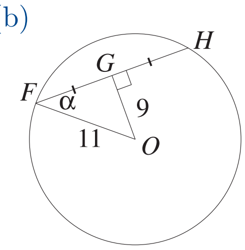
(f)


(g)


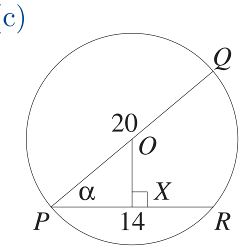
(h)


3. (a)


Find AO.

(b)


Find FH and $\cos\alpha$.

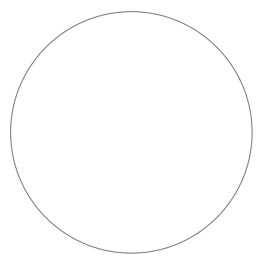
(c)


Find OX, QR and $\cos\alpha$.

4. CONSTRUCTION: *Construct the centre of a given circle.*

(a) Trace the circle drawn to the right, then use the construction given in Box 8 to find its centre.

(b) Trace it again, then use and explain this alternative construction. Construct any chord AB, and construct its perpendicular bisector — let the bisector meet the circle at P and Q, and construct the midpoint O of PQ.



5. CONSTRUCTION: *Construct the circumcircle of a given triangle.*
Place three non-collinear points towards the centre of a page, then use the construction given in Box 9 to construct the circle through these three points.

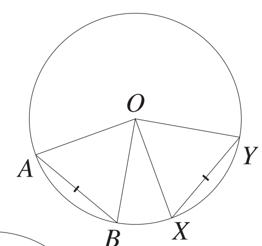
DEVELOPMENT

6. COURSE THEOREM: *Equal chords subtend equal angles at the centre, and are equidistant from the centre.*
In the diagram opposite, AB and XY are equal chords.

(a) Prove that $\triangle AOB \equiv \triangle XOY$.

(b) Prove that $\angle AOB = \angle XOY$.

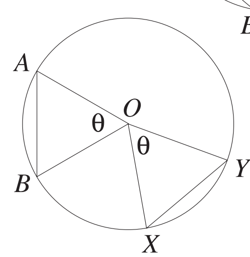
(c) Prove that the chords are equidistant from the centre.



7. COURSE THEOREM: *Two chords subtending equal angles at the centre have equal lengths.*
In the diagram opposite, the angles $\angle AOB$ and $\angle XOY$ subtended by AB and XY are equal.

(a) Prove that $\triangle AOB \equiv \triangle XOY$.

(b) Hence prove that $AB = XY$.

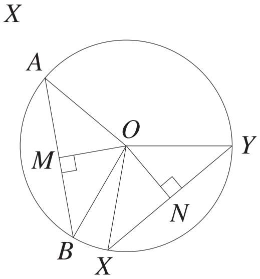


8. COURSE THEOREM: *Two chords equidistant from the centre have equal lengths.*
In the diagram opposite, $OM = ON$.

(a) Prove that $\triangle OAM \equiv \triangle OXN$.

(b) Prove that $\triangle OBM \equiv \triangle OYN$.

(c) Hence prove that $AB = XY$.



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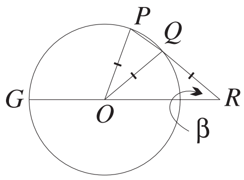
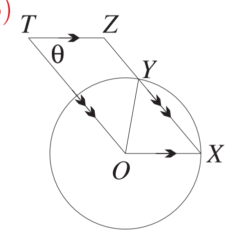
9. Two parallel chords in a circle of diameter 40 have length 20 and 10. What are the possible distances between the chords?

10. (a)



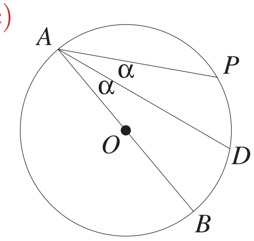
Prove that $\angle POG = 3\beta$.

(b)



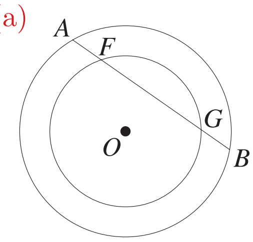
Prove that $\angle TOY = \theta$.

(c)



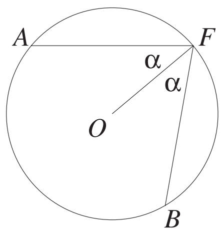
Prove that $OD \parallel AP$.

11. (a)



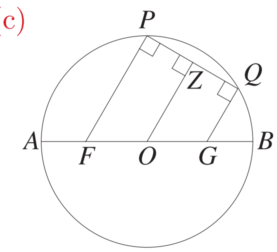
Prove that $AF = BG$.
[HINT: First prove that $\triangle OAF \equiv \triangle OBG$.]

(b)



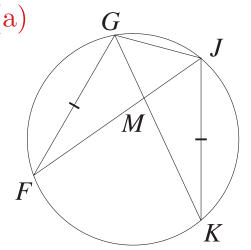
Prove that $AF = BF$.
[HINT: First prove that $\triangle OAF \equiv \triangle OBF$.]

(c)



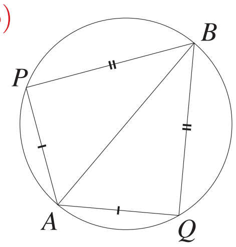
Prove that $AF = BG$.
[HINT: First use intercepts to prove that $FO = OG$.]

12. (a)



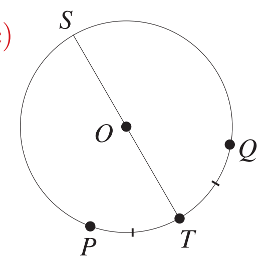
Prove that $FJ = KG$, and that $MG = MJ$.

(b)



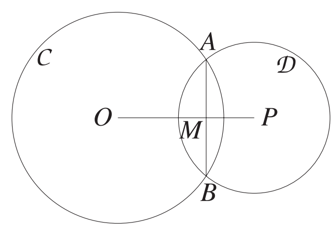
Prove that $\angle PAB = \angle QAB$, and that AB is a diameter.

(c)



Prove that $SP = SQ$, and that $PQ \perp ST$.

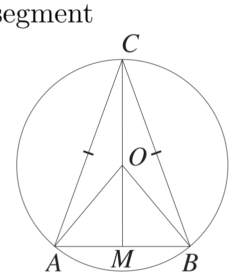
13. THEOREM: *When two circles intersect, the line joining their centres is the perpendicular bisector of the common chord.* In the diagram opposite, two circles intersect at A and B.



(a) Prove that $\triangle OAP \equiv \triangle OBP$.
(b) Hence prove that $\triangle OMA \equiv \triangle OMB$.
(c) Hence prove that $AM = MB$ and $AB \perp OP$.
(d) Under what circumstances will $OAPB$ form a rhombus?

14. In the configuration of the previous question, suppose also that each circle passes through the centre of the other (the circles will then have the same radius).
(a) Prove that the common chord subtends 120° at each centre.
(b) Find the ratio $AB : OP$. (c) Use the formula for the area of the segment to find the ratio of the overlapping area to the area of circle $\mathcal{C}$.

15. THEOREM: *If an isosceles triangle is inscribed in a circle, then the line joining the apex and the centre is perpendicular to the base.* In the diagram opposite, $CA = CB$.



(a) Prove that $\angle CAO = \angle CBO$ and $\angle ACM = \angle BCM$.
(b) Hence prove that $COM \perp AB$.

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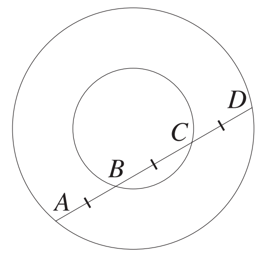
16. In the diagram to the right, the two concentric circles have radii 1 and 2 respectively.



(a) What is the length of the chord AD?

(b) How far is the chord from the centre O?

[HINT: Let $2x = AB = BC = CD$, and let h be the distance from the centre O. Then use Pythagoras' theorem.]

17. TRIGONOMETRY: A chord of length ℓ subtends an angle θ at the centre of a circle of radius r.

(a) Prove that $\ell^2 = 2r^2(1 - \cos\theta)$.

(b) Prove that $\ell = 2r \sin \frac{1}{2}\theta$.

(c) Use trigonometric identities to reconcile the two results.

18. COORDINATE GEOMETRY: Using the result of Box 9, or otherwise, find the centre and radius of the circle passing through A, B and the origin $O(0,0)$ in each case:

(a) $A = (4,0)$, $B = (4,8)$

(b) $A = (4,0)$, $B = (2,12)$

(c) $A = (4,0)$, $\triangle ABO$ equilateral

(d) $A = (6,2)$, $B = (2,6)$

EXTENSION

19. The ratio of the length of a chord of a circle to the diameter is $\lambda : 1$. The chord moves around the circle so that its length is unchanged. Explain why the locus of the midpoint M of the chord is a circle, and find the ratio of the areas of the two circles.

20. An n-sided regular polygon is inscribed in a circle. Let the ratio of the perimeter of the polygon to the circumference of the circle be $\lambda : 1$, and let the ratio of the area of the polygon to the area of the circle be $\mu : 1$.

(a) Find λ and μ for $n = 3$, 4, 6 and 8.

(b) Find expressions of λ and μ as functions of n, explain why they both have limit 1, and find the smallest value of n for which: (i) $\lambda > 0{\cdot}999$ (ii) $\mu > 0{\cdot}999$

9 B Angles at the Centre and Circumference

This section studies the relationship between angles at the centre of a circle and angles at the circumference. The converse of the angle in a semicircle theorem is new work.

Angles in a Semicircle: An *angle in a semicircle* is an angle at the circumference subtended by a diameter of the circle. Traditionally, the following theorem is attributed to the early Greek mathematician Thales, and is said to be the first mathematical theorem ever formally proven.

10 COURSE THEOREM: An angle in a semicircle is a right angle.

GIVEN: Let AOB be a diameter of a circle with centre O, and let P be a point on the circle distinct from A and B.

AIM: To prove that $\angle APB = 90^\circ$.

CONSTRUCTION: Join OP.

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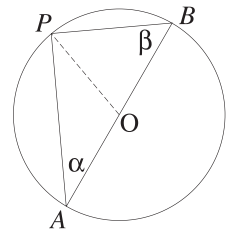


PROOF: Let $\angle A = \alpha$ and $\angle B = \beta$.
Now $OA = OP = OB$ (radii of circle),
forming two isosceles triangles $\triangle AOP$ and $\triangle BOP$,
and so $\angle APO = \alpha$ and $\angle BPO = \beta$.
But $(\alpha + \beta) + \alpha + \beta = 180^\circ$ (angle sum of $\triangle ABP$),
so $\alpha + \beta = 90^\circ$, and $\angle APB = 90^\circ$.

WORKED EXERCISE: Find α, and prove that A, O and D are collinear.

SOLUTION: First, $\angle BAC = 90^\circ$ (angle in a semicircle),
so $\alpha = 67^\circ$ (angle sum of $\triangle BAC$).
Secondly, $\angle ACD = 90^\circ$ (co-interior angles, $AB \parallel CD$),
and $\angle D = 90^\circ$, (angle in a semicircle),
so $ABCD$ is a rectangle (all angles are right angles).
Since the diagonals of a rectangle bisect each other,
the diagonal AD passes through the midpoint O of BC.

Converse of the Angle in a Semicircle Theorem: The converse theorem essentially says 'every right angle is an angle in a semicircle', so its statement must assert the existence of the semicircle, given a right triangle.

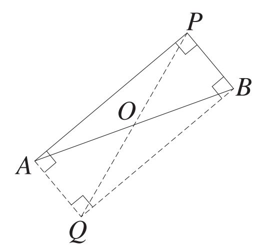
11 COURSE THEOREM: Conversely, the circle whose diameter is the hypotenuse of a right triangle passes through the third vertex of the triangle.

OR

The midpoint of the hypotenuse of a right triangle is equidistant from all three vertices of the triangle.

GIVEN: Let ABP be a triangle right-angled at P.

AIM: To prove that P lies on the circle with diameter AB.

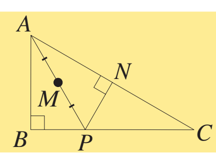


CONSTRUCTION: Complete $\triangle APB$ to a rectangle $APBQ$, and let the diagonals AB and PQ intersect at O.

PROOF: The diagonals of the rectangle $APBQ$ are equal, and bisect each other.
Hence $OA = OB = OP = OQ$, as required.

WORKED EXERCISE: From any point P on the side BC of a triangle ABC right-angled at B, a perpendicular PN is drawn to the hypotenuse. Prove that the midpoint M of AP is equidistant from B and N.

SOLUTION: Since AP subtends right angles at N and B, the circle with diameter AP passes through B and N. Hence the centre M of the circle is equidistant from B and N.



Angles at the Centre and Circumference: A semicircle subtends a straight angle at the centre, which is twice the right angle it subtends at the circumference. This relationship can be generalised to a theorem about angles at the centre and circumference standing on any arc.

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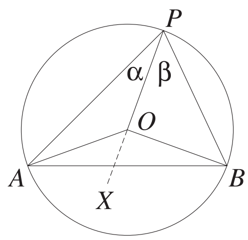
12 COURSE THEOREM: The angle subtended at the centre of a circle by an arc is twice any angle at the circumference standing on the same arc.

The angle 'standing on an arc' means the angle subtended by the chord joining its endpoints.

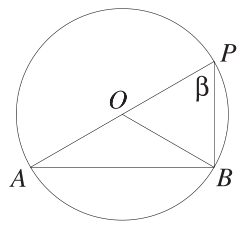
GIVEN: Let AB be an arc of a circle with centre O, and let P be a point on the opposite arc.

AIM: To prove that $\angle AOB = 2 \times \angle APB$.

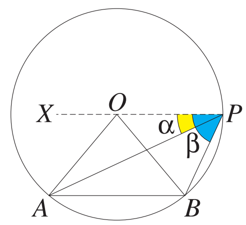
CONSTRUCTION: Join PO, and produce to X. Let $\angle APO = \alpha$ and $\angle BPO = \beta$.



Case 1



Case 2

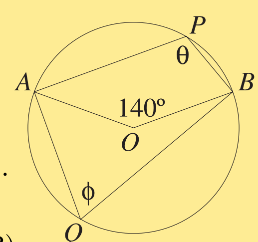


Case 3

PROOF: There are three cases, depending on the position of P.
In each case, the equal radii $OA = OP = OB$ form isosceles triangles.
CASE 1: $\angle PAO = \alpha$ and $\angle PBO = \beta$ (base angles of isosceles triangles).
Hence $\angle AOX = 2\alpha$ and $\angle BOX = 2\beta$ (exterior angles),
and so $\angle AOB = 2\alpha + 2\beta = 2(\alpha + \beta) = 2 \times \angle APB$, as required.
The other two cases are left to the exercises.

NOTE: The converse of this theorem is also true, but is not specifically in the course. It is set as an exercise in the Extension section following.

WORKED EXERCISE: Find θ and ϕ in the diagram opposite, where O is the centre of the circle.



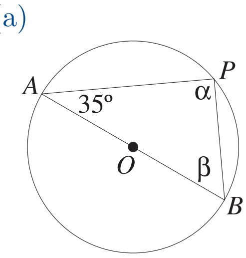
SOLUTION: First, $\phi = 70^\circ$ (angles on the same arc APB).
Secondly, reflex $\angle AOB = 220^\circ$ (angles in a revolution),
so $\theta = 110^\circ$ (angles on the same arc AQB).

Exercise 9B

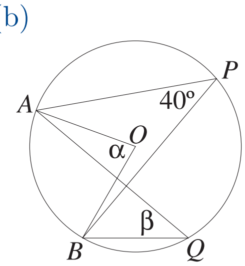
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, points labelled O or Z are centres of the appropriate circles.

1. Find α, β, γ and δ in each diagram below.

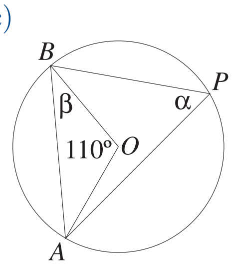
(a)



(b)



(c)



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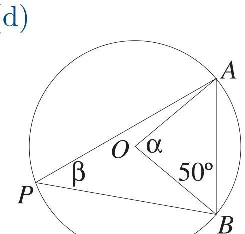

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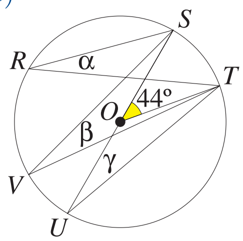
(e)


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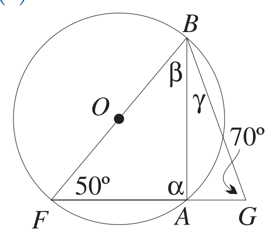

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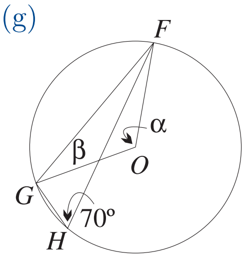

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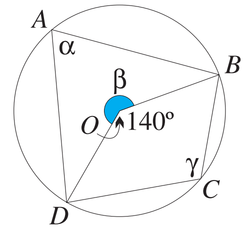
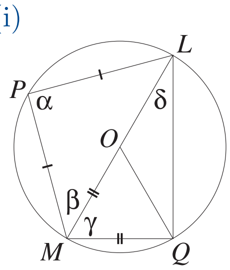


(i)


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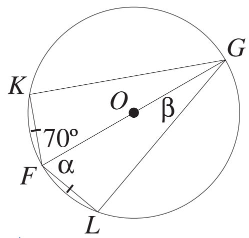

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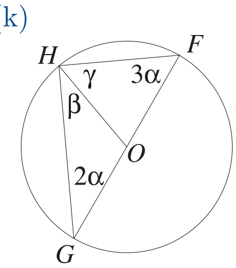
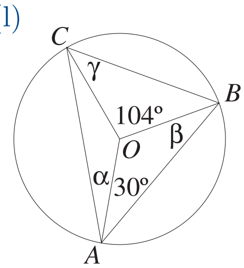
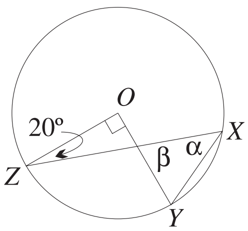
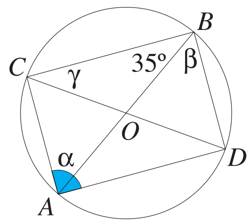


(l)


(m)


(n)


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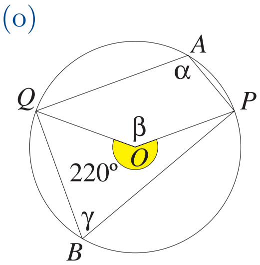

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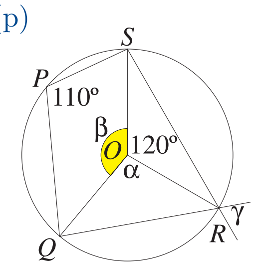
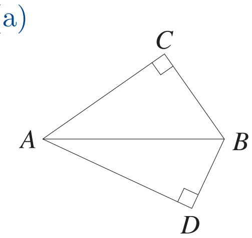


2. In each diagram, name a circle containing four points, and name a diameter of it. Give reasons for your answers.

(a)


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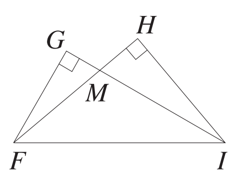

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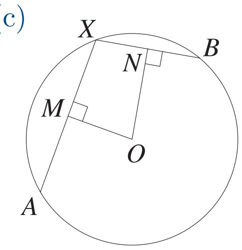

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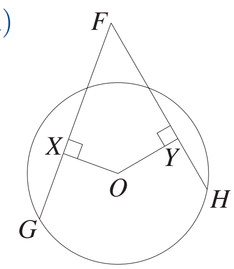
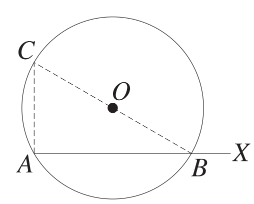


3. A photographer is photographing the façade of a building. To do this effectively, he has to position himself so that the two ends of a building subtend a right angle at his camera. Describe the locus of his possible positions, and explain why he must be a constant distance from the midpoint of the building.

4. CONSTRUCTION: *Constructing a right angle at the end-point of an interval.* Let AX be an interval. With any centre O above or below the interval AX, construct a circle with radius OA. Let the circle pass through AX again at B. Construct the diameter through B, and let it meet the circle again at C. Prove that $AC \perp AX$.



DEVELOPMENT

5. COURSE THEOREM: Complete the other two cases of the proof that the angle at the centre subtended by an arc is twice the angle at the circumference subtended by that arc.

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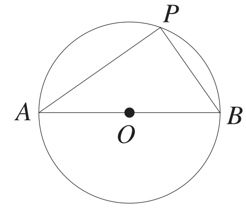
6. ALTERNATIVE PROOFS THAT AN ANGLE IN A SEMICIRCLE IS A RIGHT ANGLE: Let AB be a diameter of a circle with centre O, and let P be any other point on the circle.

(a) Euclid's proof, Book 1, Proposition XX: Produce AP to Q, and join OP. Let $\angle A = \alpha$ and $\angle B = \beta$.

(i) Explain why $\angle QPB = \alpha + \beta$ and $\angle APB = \alpha + \beta$.

(ii) Hence prove that $\angle APB$ is a right angle.



(b) Proof using rectangles: Join PO and produce it to the diameter POR. Use the diagonal test to prove that $APBR$ is a rectangle, and hence that $\angle APB = 90°$.

(c) Proof using intercepts: Let M be the midpoint of AP. Explain why $OM \perp AP$ and $OM \parallel BP$. Hence prove that $\angle P$ is a right angle.

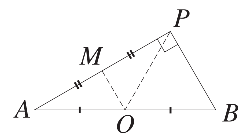
7. ALTERNATIVE PROOFS OF THE CONVERSE: Let $\triangle ABP$ be right-angled at P.

(a) Proof using intercepts: Let O and M be the midpoints of AB and AP respectively.

(i) Prove that $OM \perp AP$.

(ii) Prove that $\triangle AOM \equiv \triangle POM$.

(iii) Explain why O is equidistant from A, B and P.

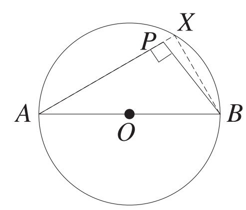


(b) A proof using the forward theorem: Construct the circle with diameter AB. Let AP (produced if necessary) meet the circle again at X. We must prove that the points P and X coincide.

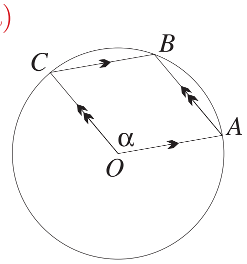
(i) Explain why $\angle AXB = 90°$.

(ii) Explain why $PB \parallel XB$.

(iii) Explain why the points P and X coincide.

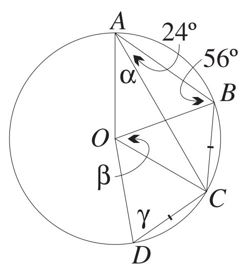


8. (a)



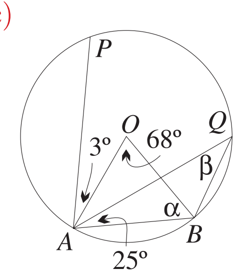
Explain why $\angle B = \alpha$, and find reflex $\angle O$. Then prove that $\alpha = 120°$.

(b)



Find α, β and γ.

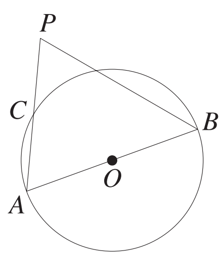
(c)



Find α and β. Then prove that $AP \parallel BQ$.

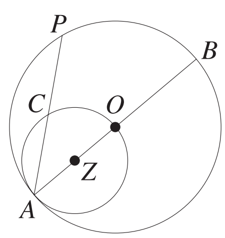
9. In each case, prove that C is the midpoint of AP. In part (a), $AB = PB$.

(a)



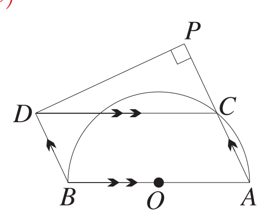
[HINT: Join BC.]

(b)



[HINT: Join OC and PB.]

(c)



[HINT: Join BC.]

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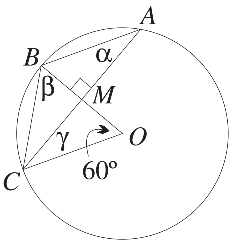
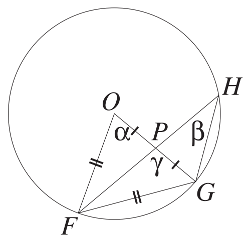
10. Give careful arguments to find α, β and γ in each diagram. In part (a), prove also that $OM = MB$. [HINT: Parts (b) and (c) will need congruence.]

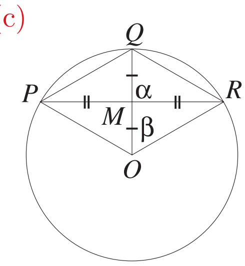
(a)



(b)

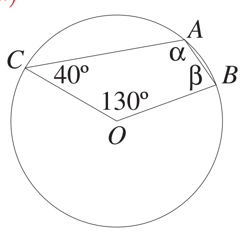


(c)

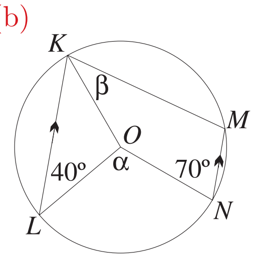


11. Find α, β, γ and δ in each diagram. Begin part (c) by proving that $\alpha = 120°$.

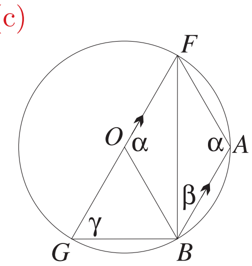
(a)



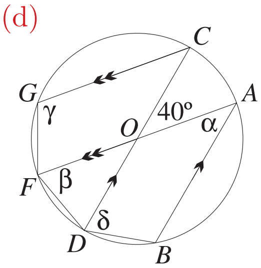
(b)



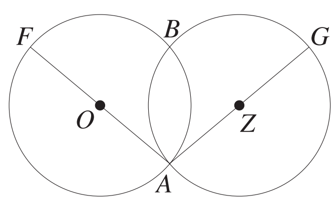
(c)



(d)



12. (a)



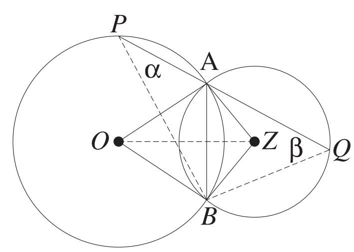
AOF and AZG are both diameters.

(i) Join AB, and hence prove that $\angle ABF = \angle ABG = 90°$.

(ii) Show that the points F, B and G are collinear.

(iii) If the radii are equal, prove that $FB = BG$.

(b)



A line through A meets the two circles again at P and Q. Let $\angle P = \alpha$ and $\angle Q = \beta$.

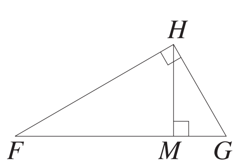
(i) Prove that $\triangle AOZ \equiv \triangle BOZ$.

(ii) Prove that OZ bisects $\angle AOB$ and $\angle AZB$.

(iii) Prove that $\angle BOZ = \alpha$ and $\angle BZO = \beta$.

(iv) Prove that $\angle PBQ = \angle OBZ$.

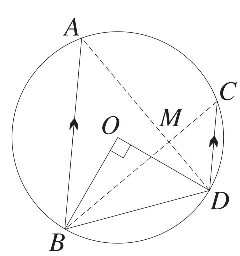
13. (a)



(i) Prove that the circles FMH, HMG and GHF have diameters FH, HG and GF respectively.

(ii) Prove that the sum of the areas of the circles FMH and GMH equals the area of the circle FHG.

(b)



(i) Prove that $\angle A = \angle C = 45°$.

(ii) Prove that $AD \perp BC$.

(iii) Prove that M lies on the circle BDO.

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EXTENSION

14. MINIMISATION: In a rectangle inscribed in a circle, let length : breadth $= \lambda : 1$.

(a) Show that the ratio of the areas of the circle and the rectangle is $\dfrac{\pi}{4}\left(\lambda + \dfrac{1}{\lambda}\right)$.

(b) Prove that the ratio of the areas has its minimum when the rectangle is a square, and find this minimum ratio.

(c) Find λ when the ratio of the areas is twice its minimum value.

15. THEOREM: *The converse of the angle at the centre and circumference theorem.* Use the method of question 7(c) to prove that if $\triangle AOB$ is isosceles with apex O, and a point P lies on the same side of AB as O such that $\angle AOB = 2\angle APB$, then the circle with centre O and radius $OA = OB$ also passes through C.

16. CIRCULAR MOTION: A horse is travelling around a circular track at a constant rate, and a punter standing at the edge of the track is following him with binoculars. Use circle geometry to prove that the punter's binoculars are rotating at a constant rate.

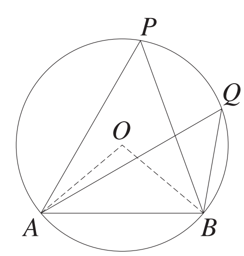
9 C Angles on the Same and Opposite Arcs

The previous theorem relating angles at the centre and circumference has two important consequences. First, any two angles on the same arc are equal. Secondly, two angles in opposite arcs are supplementary, or alternatively, the opposite angles of a cyclic quadrilateral are supplementary,

Angles at the Circumference Standing on the Same Arc: An angle subtended by an arc at the circumference of a circle is also called 'an angle in a segment', just as an angle in a semicircle is called 'an angle in a semicircle'. This accounts for the alternative statement of the theorem:

13 COURSE THEOREM: Two angles in the same or equal segments are equal.
OR
Two angles at the circumference standing on the same or equal arcs are equal.

The proof of this theorem relates the two angles at the circumference back to the single angle at the centre (the case of 'equal arcs' is left to the reader):



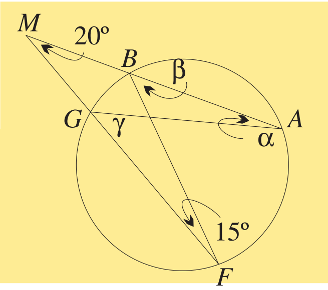
GIVEN: Let AB be an arc of a circle with centre O, and let P and Q be points on the opposite arc.

AIM: To prove that $\angle APB = \angle AQB$.

CONSTRUCTION: Join AO and BO.

PROOF: $\angle AOB = 2 \times \angle APB$ (angles on the same arc AB),
and $\angle AOB = 2 \times \angle AQB$ (angles on the same arc AB).
Hence $\angle APB = \angle AQB$.

WORKED EXERCISE: Find α, β and γ in the diagram opposite.



SOLUTION: $\alpha = 15^\circ$ (angles on the same arc BG),
$\beta = 35^\circ$ (exterior angle of $\triangle BFM$),
$\gamma = 35^\circ$ (angles on the same arc AF).

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Cyclic Quadrilaterals: A *cyclic quadrilateral* is a quadrilateral whose vertices lie on a circle (we say that the quadrilateral is *inscribed* in the circle). A cyclic quadrilateral is therefore formed by taking two angles standing on opposite arcs, which is why its study is relevant here.

14 COURSE THEOREM:
- Opposite angles of a cyclic quadrilateral are supplementary.
- An exterior angle of a cyclic quadrilateral equals the opposite interior angle.

GIVEN: Let $ABCD$ be a cyclic quadrilateral, with side BC produced to T, and let O be the centre of the circle $ABCD$. Let $\angle A = \alpha$ and $\angle C = \gamma$.

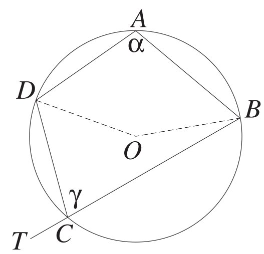
AIM: To prove: (a) $\alpha + \gamma = 180°$ (b) $\angle DCT = \alpha$

CONSTRUCTION: Join BO and DO.

PROOF: There are two angles at O, one reflex, one non-reflex.

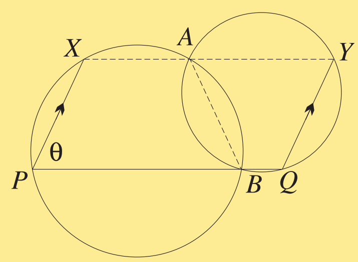
(a) Taking angles on the arc BCD, $\angle BOD = 2\alpha$ (facing C),
Taking angles on the arc BAD, $\angle BOD = 2\gamma$ (facing A).
Hence $2\alpha + 2\gamma = 360°$ (angles in a revolution),
so $\alpha + \gamma = 180°$, as required.

(b) Also, $\angle DCT = 180° - \gamma$ (straight angle),
$= \alpha$, by part (a).



WORKED EXERCISE: In the diagram below, prove that X, A and Y are collinear.

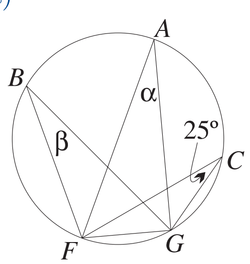
SOLUTION: Join AB, AX and AY, and let $\angle P = \theta$.
$\angle XAB = 180° - \theta$
(opposite angles of cyclic quadrilateral $ABPX$).
Also $\angle Q = 180° - \theta$
(co-interior angles, $PX \parallel QY$),
so $\angle YAB = \theta$
(opposite angles of cyclic quadrilateral $ABQY$).
Hence $\angle XAY = 180°$, and so XAB is a straight line.

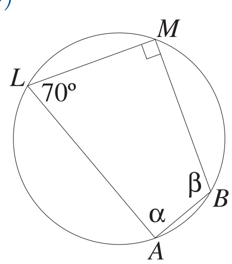


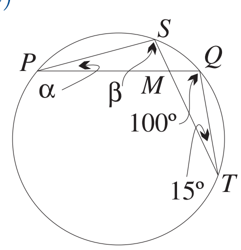
Exercise 9C

NOTE: In each question, all reasons must always be given. Unless otherwise indicated, points labelled O or Z are centres of the appropriate circles.

1. Find α, β and γ as appropriate in each diagram below.

(a)


(b)


(c)


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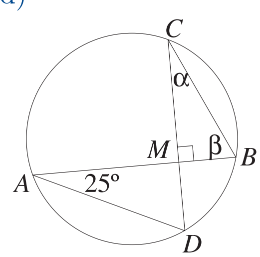

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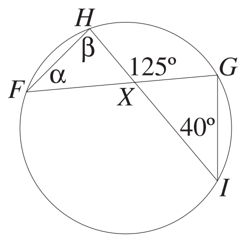
(e)



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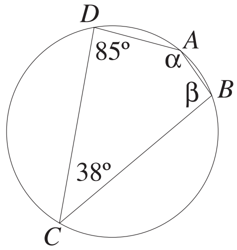
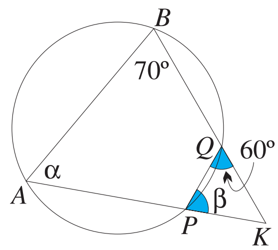
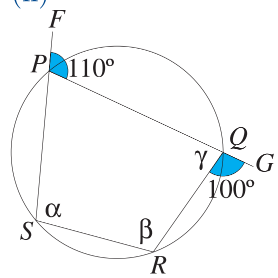


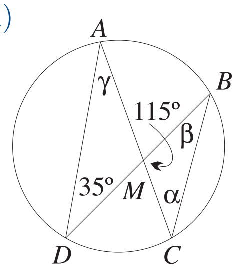
(g)



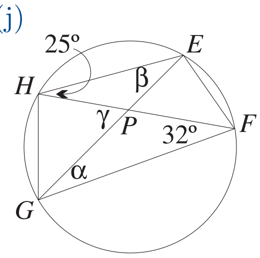
(h)



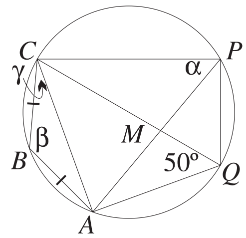
(i)



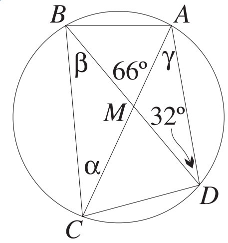
(j)



(k)

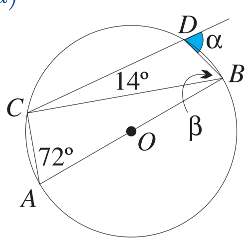


(l)

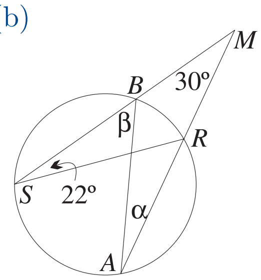


2. Find α, β and γ as appropriate in each diagram.

(a)



(b)



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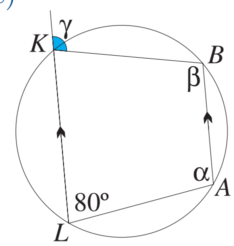

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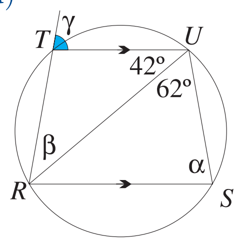

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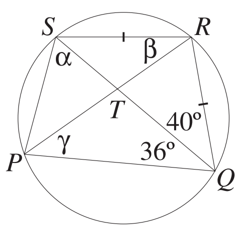

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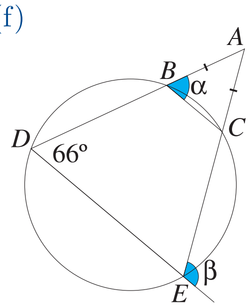

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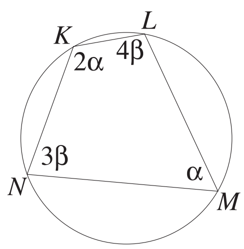

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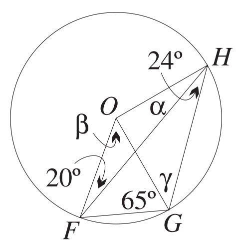
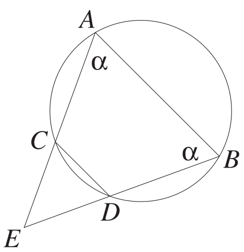


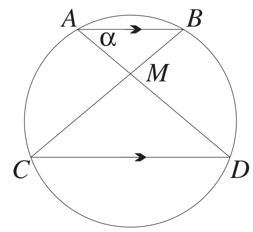
3. Suppose that $ABCD$ is a cyclic quadrilateral. Draw a diagram of $ABCD$, and then explain why $\sin A = \sin C$ and $\sin B = \sin D$.

4. (a)



Prove that $CD \parallel AB$.
Prove that $EC = ED$.

(b)



Prove that $\angle A = \angle B = \angle C = \angle D$.
Prove that $AD = BC$.

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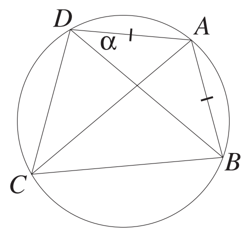
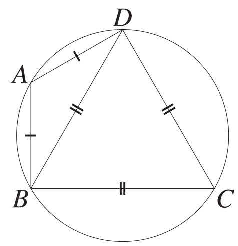
(c)



Prove that $\angle ACB = \alpha$.
Prove that AC bisects $\angle DCB$.

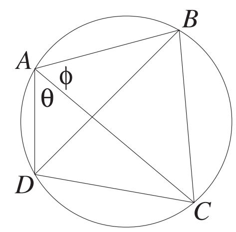
(d)



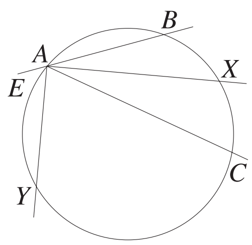
Prove that $AB \perp BC$.

DEVELOPMENT

5. ALTERNATIVE PROOF THAT THE OPPOSITE ANGLES OF A CYCLIC QUADRILATERAL ARE SUPPLEMENTARY:
In the diagram opposite:
(a) Prove that $\angle DBC = \theta$ and $\angle BDC = \phi$.
(b) Hence prove that $\angle DAB$ and $\angle DCB$ are supplementary.

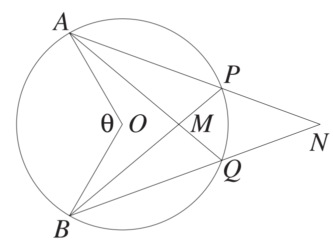


6. (a)



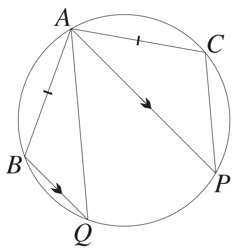
AX bisects $\angle CAB$, AY bisects $\angle CAE$.
Prove that $\angle YAX = 90°$.
Prove that $\angle YCX = 90°$.

(b)



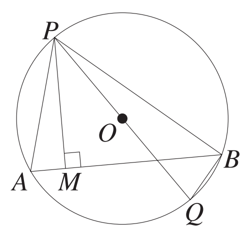
Give a reason why $\angle APB = \frac{1}{2}\theta$.
Show that $\angle BPN = \angle AQN = 180° - \frac{1}{2}\theta$.
Show that $\angle AMB + \angle ANB = \theta$.

(c)



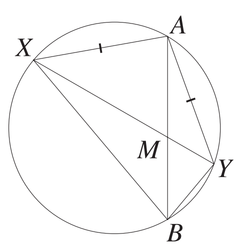
Give a reason why $\angle Q = \angle P$.
Prove that $AQ \parallel CP$.

(d)



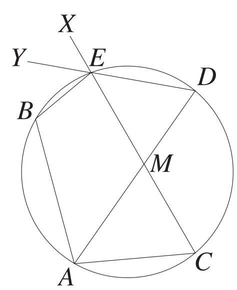
Give a reason why $\angle A = \angle Q$.
Prove that $\angle APM = \angle QPB$.

(e)



Give a reason why $\angle BXY = \angle BAY$.
Prove that AB bisects $\angle XBY$.
Prove that $\angle XMB = \angle AYB$.

(f)



Give a reason why $\angle BAD = \angle BEY$.
Given that DA bisects $\angle BAC$,
prove that YE bisects $\angle XEB$.

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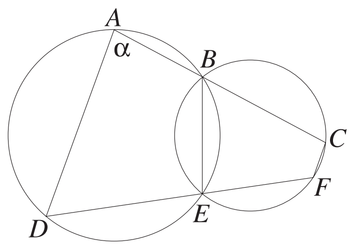
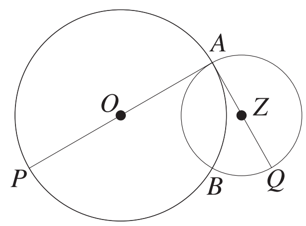
7. (a)



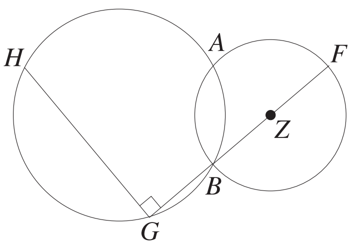
Prove that $\angle BEF = \alpha$ and find $\angle C$.
Prove that $AD \parallel CF$.

(b)



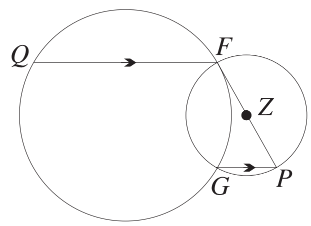
Find $\angle ABP$ and $\angle ABQ$.
Prove that P, B and Q are collinear.

(c)



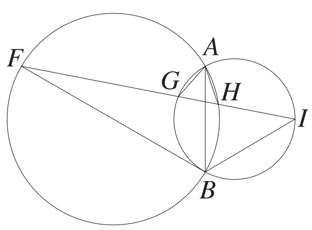
Find $\angle BAF$ and $\angle BAH$.
Prove that H, A and F are collinear.

(d)



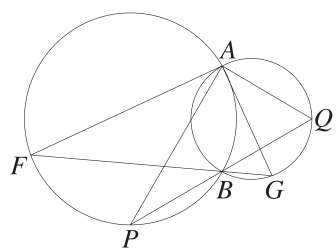
Prove that QG is a diameter. If the radii are equal, prove that $QG \parallel FP$.

(e)



Give a reason why $\angle FBA = \angle FHA$.
Given that AB bisects $\angle FBI$,
prove that $AG = AH$.

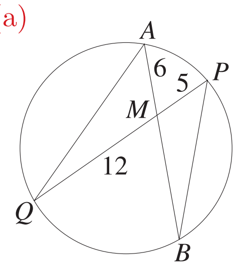
(f)



Give a reason why $\angle Q = \angle G$.
Given that FBG and PBQ are straight lines, prove that $\angle FAP = \angle GAQ$.

8. In each diagram, prove that $\triangle AMQ \,|||\, \triangle PMB$. Then find MB.

(a)



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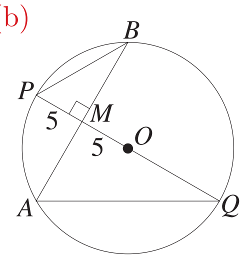

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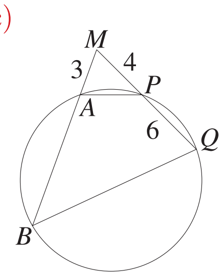

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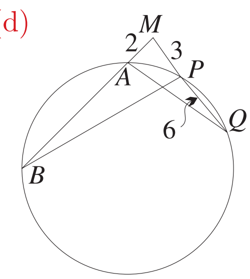


9. THEOREM: *Let the two pairs of opposite sides of a cyclic quadrilateral meet, when produced, at X and Y respectively. Then the angle bisectors of $\angle X$ and $\angle Y$ are perpendicular.*

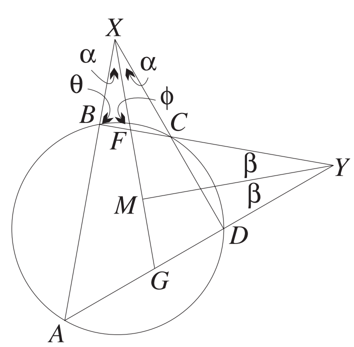
In the diagram opposite:

(a) Explain why $\angle XDA = \theta$.

(b) Using $\triangle XGD$ and $\triangle XFB$, prove that $\angle XGD = \phi$.

(c) Using $\triangle MYF$ and $\triangle MYG$, prove that $YM \perp XM$.

(d) How should this theorem be restated when a pair of opposite sides is parallel?



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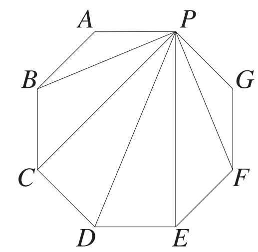
10. THEOREM: *Diagonals in a regular polygon.*

(a) In the regular octagon opposite, use the circumcircle to prove that the six angles between adjacent diagonals at P are all equal. Hence find the value of $\alpha = \angle APB$.

(b) More generally, prove that the angles between adjacent diagonals at any vertex of an n-sided regular polygon are all equal, and have the value $\dfrac{180°}{n}$.



11. (a) Prove that a cyclic parallelogram is a rectangle.

(b) Prove that a cyclic rhombus is a square.

(c) Prove that the non-parallel opposite sides of a cyclic trapezium are equal.

12. Let A, B, C, D, and E be five points in order around a circle with centre O, and let AOE be a diameter. Prove that $\angle ABC + \angle CDE = 270°$.

13. (a) Prove that if two chords of a circle bisect each other, then they are both diameters.

(b) Prove that if the chords AB and PQ intersect at M and $MA = MP$, then $MB = MQ$, $BP = AQ$ and $AP \parallel QB$.

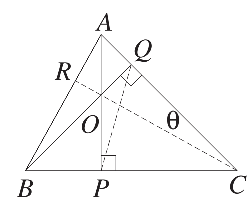
14. THE ORTHOCENTRE THEOREM: *The three altitudes of a triangle are concurrent (their intersection is called the* orthocentre *of the triangle).*

In the diagram opposite, the two altitudes AP and BQ meet at O. Join CO and produce it to R, and join PQ.

(a) Explain why $OPCQ$ and $AQPB$ are cyclic.

(b) Let $\angle ACR = \theta$, and explain why $\angle APQ = \angle ABQ = \theta$.

(c) Use $\triangle OQC$ and $\triangle ORB$ to prove that $CR \perp AB$.



15. THE SINE RULE AND THE CIRCUMCIRCLE: *The ratio of any side of a triangle to the sine of the opposite angle is the diameter of the circumcircle.*

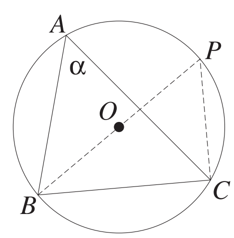
Let $\angle A$ in $\triangle ABC$ be acute, and let O be the centre of the circumcircle of $\triangle ABC$. Join BO and produce it to a diameter BOP, then join PC.

(a) Let $\angle A = \alpha$, and explain why $\angle P = \alpha$.

(b) Explain why $\triangle BPC$ is a right triangle.

(c) Hence prove that $\dfrac{BC}{\sin\alpha} = BOP$.

(d) Repeat the construction and proof when $\angle A$ is obtuse.



EXTENSION

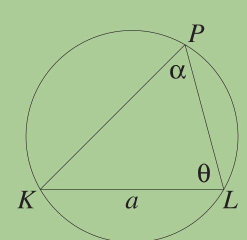
16. MAXIMISATION: In the diagram below, KL is a fixed chord of length a, and the point P varies on the major arc KL. Let y be the sum of the lengths of PK and PL.

(a) Explain why α is constant as P varies.

(b) Use the sine rule to prove that $y = \dfrac{a}{\sin\alpha}\big(\sin\theta + \sin(\theta+\alpha)\big)$.

(c) Find $\dfrac{dy}{d\theta}$, and show that $\dfrac{d^2y}{d\theta^2} = -y$.

(d) Prove that y is maximum when $\theta = \frac{1}{2}(\pi - \alpha)$, then find and simplify the maximum value.



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17. MATHEMATICAL INDUCTION: *The alternating sums of the angles of a cyclic polygon.*

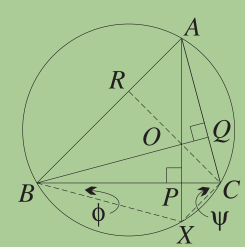
(a) Prove that if $ABCD$ is a cyclic quadrilateral, then $\angle A - \angle B + \angle C - \angle D = 0$.

(b) Prove that if $A_1A_2A_3A_4A_5A_6$ is a cyclic hexagon, then $\displaystyle\sum_{k=1}^{6}(-1)^k \angle A_k = 0$.

[HINT: Use the major diagonal A_1A_4 to divide the hexagon into two cyclic quadrilaterals, then apply part (a) to each quadrilateral.]

(c) Use mathematical induction, and the same method as in part (b), to prove that for any cyclic polygon $A_1A_2\ldots A_{2n}$ with an even number of vertices, $\displaystyle\sum_{k=1}^{2n}(-1)^k \angle A_k = 0$.

18. THE ORTHOCENTRE THEOREM: *A proof using the circumcircle.* In the diagram, the two altitudes AP and BQ meet at O. Join CO and produce it to R. Produce AP to meet the circumcircle of $\triangle ABC$ at X, and join BX and CX. Let $\angle CBX = \phi$ and $\angle BCX = \psi$.



(a) Explain why $\angle CAX = \phi$.

(b) Using $\triangle QOA$ and $\triangle POB$, prove that $\angle PBO = \phi$.

(c) Prove that $\triangle POB \equiv \triangle PXB$, and hence that $PO = PX$.

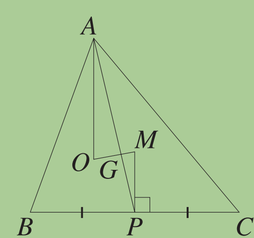
(d) Prove that $\triangle POC \equiv \triangle PXC$, and hence that $\angle OCP = \psi$.

(e) By comparing $\triangle POC$ and $\triangle ROA$, prove that $CR \perp AB$.

19. The last two questions of Exercise 4J in the Year 11 volume contain a variety of algebraic results about cyclic quadrilaterals and their circumcircles, established using trigonometry. Those results and their proofs could be examined in the present context of Euclidean geometry. See also the related questions about the circumcircle and incircle of a triangle at the end of Exercises 4H and 4I in the Year 11 volume.

20. THE EULER LINE THEOREM: *The orthocentre, centroid and circumcentre of a triangle are collinear (the line is called the* Euler line*), with the centroid trisecting the interval joining the other two centres.*

Let M and G be the circumcentre and centroid respectively of $\triangle ABC$. Join MG, and produce it to a point O so that $OG : GM = 2 : 1$. We must prove that O is the orthocentre of $\triangle ABC$.



(a) Let P be the midpoint of BC. Use the fact that $AG : GP = 2 : 1$ to prove that $\triangle GMP \,|||\, \triangle GAO$.

(b) Hence prove that O lies on the altitude from A.

(c) Complete the proof.

9 D Concyclic Points

A set of points is called *concyclic* if they all lie on a circle. The converses of the two theorems of the previous section provide two general tests for four points to be concyclic. There is an important logical structure here to keep in mind. First, any two distinct points lie on a unique line, but three points may or may not be collinear. Secondly, any three non-collinear points are concyclic, as proven in Section 9A, but four points may or may not be concyclic.

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Concyclicity Test — Two Points on the Same Side of an Interval: We have proven that angles at the circumference standing on the same arc of a circle are equal. The converse of this is:

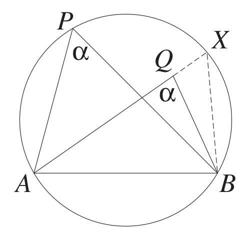
15 COURSE THEOREM: If two points lie on the same side of an interval, and the angles subtended at these points by the interval are equal, then the two points and the endpoints of the interval are concyclic.

The most satisfactory proof makes use of the forward theorem.

GIVEN: Let P and Q be points on the same side of an interval AB such that $\angle APB = \angle AQB = \alpha$.

AIM: To prove that the points A, B, P and Q are concyclic.

CONSTRUCTION: Construct the circle through A, B and P, and let the circle meet AQ (produced if necessary) at X. Join XB.



PROOF: Using the forward theorem,

$\angle AXB = \angle APB = \alpha$ (angles on the same arc AB).

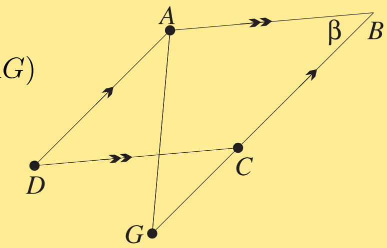
Hence $\angle AXB = \angle AQB$,

so $QB \parallel XB$ (corresponding angles are equal).

But QB and XB intersect at B, and are therefore the same line.

Hence Q and X coincide, and so Q lies on the circle.

WORKED EXERCISE: In the diagram opposite, $AB = AG$. Prove that $ACGD$ is cyclic, and that $\angle ACD = \angle AGD$.



SOLUTION: Let $\angle B = \beta$.

Then $\angle AGB = \beta$ (base angles of isosceles $\triangle BAG$)

and $\angle ADC = \beta$

(opposite angles of parallelogram $ABCD$),

so the quadrilateral $ACGD$ is cyclic,

because AC subtends equal angles at D and G.

Hence $\angle ACD = \angle AGD$ (angles on the same arc AD).

Concyclicity Test — Cyclic Quadrilaterals: The converses of the two forms of the cyclic quadrilateral theorem are:

16 COURSE THEOREM:
- If one pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is cyclic.
- If one exterior angle of a quadrilateral is equal to the opposite interior angle, then the quadrilateral is cyclic.

Since the exterior angle and the adjacent interior angle are supplementary, being angles in a straight angle, we need only prove the first test, and the second will follow immediately. The proof of the first test is similar to the previous proof, and is left to the exercises.

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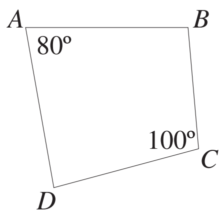
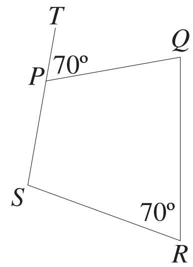
WORKED EXERCISE: Give reasons why each quadrilateral below is cyclic.

(a) 

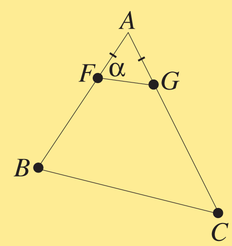
(b) 

$ABCD$ is a cyclic quadrilateral (opposite angles are supplementary).

$PQRS$ is a cyclic quadrilateral (exterior angle equals opposite interior angle).

NOTE: When the angles subtended by the interval are right angles, the four points are concyclic by the earlier theorem that a right angle was an angle in a semicircle, moreover the interval is then a diameter of the circle. These two tests for the concyclicity of four points should therefore be seen as generalisations of that theorem.

WORKED EXERCISE: Prove that if $FGCB$ is cyclic, then $FB = GC$.
Prove that if $FB = GC$, then $FGCB$ is cyclic.

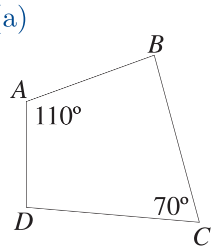


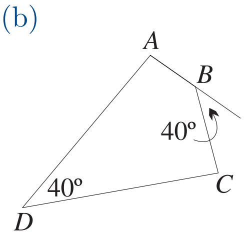
SOLUTION: Let $\angle AFG = \alpha$.
Then $\angle AGF = \alpha$ (base angles of isosceles $\triangle AFG$).
Suppose first that $FGCB$ is cyclic.
Then $\angle C = \alpha$ (exterior angle of cyclic quadrilateral $FGCB$)
and $\angle B = \alpha$ (exterior angle of cyclic quadrilateral $FGCB$),
so $AB = AC$ (opposite angles of $\triangle ABC$ are equal),
hence $FB = GC$ (subtracting the equal intervals AF and AG).
Suppose secondly that $FB = GC$.
Then $FG \parallel BC$ (intercepts on AB and AC),
so $\angle B = \alpha$ (corresponding angles, $FG \parallel BC$),
hence $FGCB$ is cyclic (exterior angle $\angle AGF$ equals interior opposite angle $\angle B$).

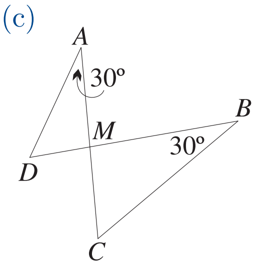
Exercise 9D

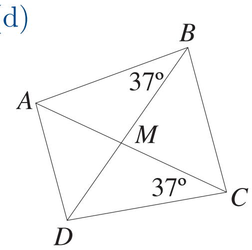
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, points labelled O are centres of the appropriate circles.

1. In each diagram, give a reason why $ABCD$ is a cyclic quadrilateral.

(a) 

(b) 

(c) 

(d) 

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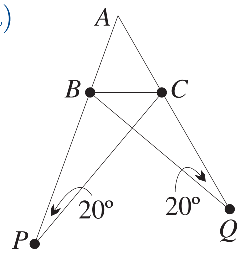
2. In each diagram, prove that the four darkened points are concyclic.

(a)


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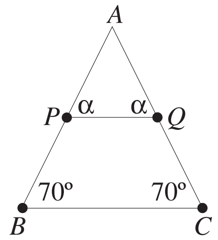

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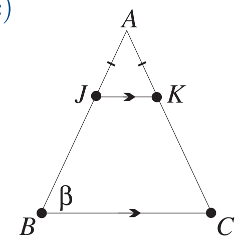

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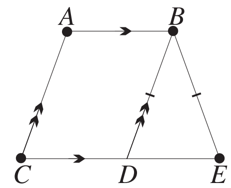

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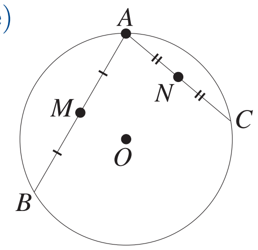

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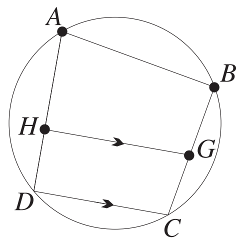

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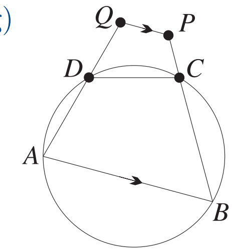

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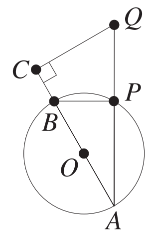

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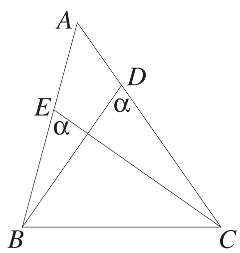
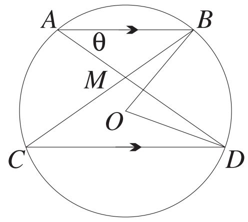


Prove that $BEDC$ is cyclic.
Hence prove that $\angle EBD = \angle ECD$,
and that $\angle ADE = \angle ABC$.

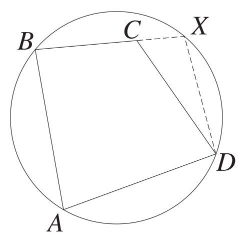
(b)


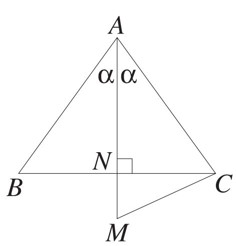
Prove that $\angle BMD = 2\theta$, and
hence prove that $BMOD$ is cyclic.
Hence prove that $\angle MBO = \angle MDO$.

4. (a) Prove that every rectangle is cyclic.
(b) Prove that any quadrilateral $ABCD$ in which $\angle A - \angle B + \angle C - \angle D = 0°$ is cyclic.

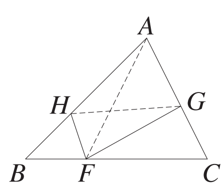
DEVELOPMENT

5. Course theorem: *If one pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is cyclic.*
Let $ABCD$ be a quadrilateral in which $\angle A + \angle BCD = 180°$.
Construct the circle through A, B and D, and let it meet BC (produced if necessary) at X. Join DX.
(a) Prove that $\angle BXD + \angle A = 180°$.
(b) Prove that $CD \parallel XD$, and that C and X coincide.



6. (a)


(i) Prove that if $ABMC$ is cyclic, then $MC \perp AC$.
(ii) Prove that if $MC \perp AC$, then $ABMC$ is cyclic.

(b)


(i) Prove that if $\angle BHF = \angle AGF$, then $FGAH$ is cyclic and $\angle AHG = \angle AFG$.
(ii) Prove that if $\angle AHG = \angle AFG$, then $FGAH$ is cyclic and $\angle BHF = \angle AGF$.

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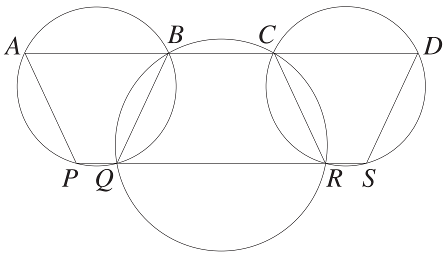
7. (a)

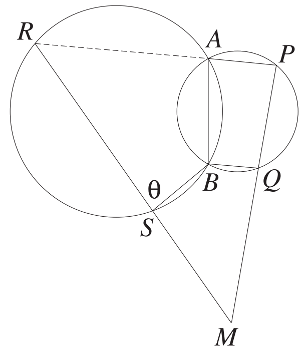


In the diagram above, $ABCD$ and $PQRS$ are straight lines, not necessarily parallel.

(i) Show that $AP \parallel CR$.

(ii) Show that $APSD$ is cyclic.

(b)



(i) Prove that $\angle PAB = \theta$.

(ii) Prove that if S, B, Q and M are concyclic, then R, A, and P are collinear.

(iii) Prove that if R, A and P are collinear, then $SBQM$ is cyclic.

8. Let AB and XY be parallel intervals, with AY and BX meeting at M.

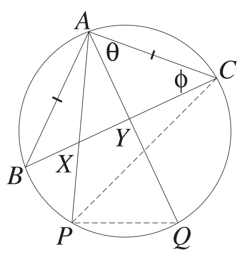
(a) Prove that if $AXYB$ is cyclic, then $MA = MB$.

(b) Prove that if $MA = MB$, then $AXYB$ is cyclic.

9. Let P, Q and R be the midpoints of three chords MA, MB and MC of a circle.

(a) Prove that $PQ \parallel AB$ and $QR \parallel BC$. (b) Prove that M, P, Q and R are concyclic.

10. (a)



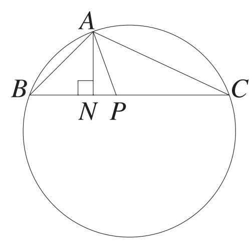
In the diagram above, $AB = AC$.

(i) Prove that $\angle CPQ = \theta$.

(ii) Prove that $\angle CPA = \phi$.

(iii) Hence prove that $PQYX$ is cyclic.

(b)



(i) Prove that if AP produced is a diameter of circle ABC, then $\angle BAN = \angle CAP$.

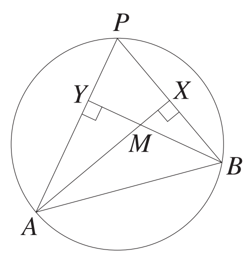
(ii) Prove that if $\angle BAN = \angle CAP$, then AP produced is a diameter of circle ABC.

11. The chord AB of the circle opposite is fixed, and the point P varies on the major arc of the circle. The altitudes AX and BY of $\triangle ABP$ meet at M.

(a) Let $\angle P = \alpha$. Explain why α is constant.

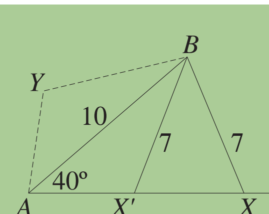
(b) Explain why $PXMY$ is cyclic.

(c) Show that $\angle AMB = 180° - \alpha$, and find the locus of M.



EXTENSION

12. TRIGONOMETRY: *The spurious ASS congruence test can be related to cyclic quadrilaterals.*
The unbroken lines represent a construction of the two possible triangles ABX and ABX' in which $\angle BAX = 40°$, $AB = 10$ and $BX = 7$. The broken lines represent $\triangle ABX'$ reflected about AB to $\triangle ABY$. Prove that the two triangles together form a cyclic quadrilateral $AXBY$.



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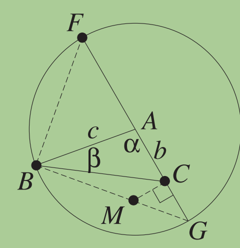
13. A TRIGONOMETRIC THEOREM: $\tan(B + \frac{1}{2}A) = \dfrac{c+b}{c-b}\tan\frac{1}{2}A$ *in any triangle ABC.*

Let ABC be a triangle in which $c > b$. Let $\angle ABC = \beta$ and $\angle CAB = \alpha$. Construct the circle with centre A passing through B, and construct the diameter $FACG$. Let the perpendicular to $FACG$ through C meet BG at M.



(a) Explain why $AF = c$ and $CG = c - b$.

(b) Prove that C, M, B and F are concyclic.

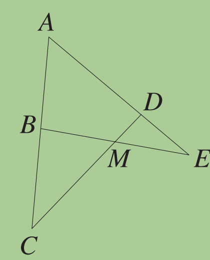
(c) Prove that $\angle BFC = \frac{1}{2}\alpha = \angle CMG$.

(d) Prove that $\angle FBC = \beta + \frac{1}{2}\alpha = \angle FMC$.

(e) Prove that $CM = (c - b)\cot\frac{1}{2}\alpha = (c + b)\cot(\beta + \frac{1}{2}\alpha)$.

(f) Adapt the construction to prove the theorem when $c < b$.

14. ABC and ADE are any two intervals meeting at A. Let BE and DC meet at M, and let the circles CMB and EMD meet again at N. Prove that $ADNC$ and $ABNE$ are cyclic. [HINT: Join NM, NC and NE.]



15. Referring to the diagram in question 11, where the chord AB is constant and P varies:

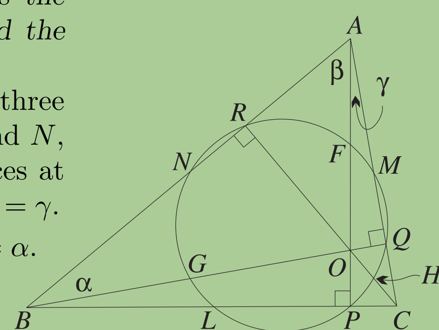
(a) Explain why $AYXB$ is cyclic, and locate the centre of this circle.

(b) Prove that $\angle YAX$ is constant, and that the interval XY has constant length.

(c) What is the locus of the midpoint of XY?

16. THE NINE-POINT CIRCLE THEOREM: *The circle through the feet of the three altitudes of a triangle passes through the three midpoints of the sides, and bisects the three intervals joining the orthocentre to the vertices. Its centre is the midpoint of the interval joining the circumcentre and the orthocentre.*

In $\triangle ABC$ opposite, P, Q and R are the feet of the three altitudes. The circle PQR meets the sides at L, M and N, and the intervals joining the orthocentre to the vertices at F, G and H. Let $\angle ABO = \alpha$, $\angle BAO = \beta$ and $\angle CAO = \gamma$.



(a) Prove that $\angle RBO = \angle RPO = \angle QPO = \angle QCO = \alpha$.

(b) Proceed similarly with β and γ.

(c) Prove that $\alpha + \beta + \gamma = 180°$.

(d) Prove that $\angle RLQ = 2\alpha$, and hence that $BL = LC$.

(e) Prove that $\angle LHC = \angle RPL$, and hence that $OH = HC$.

17. Let $ABCD$ be a square, and let P be a point such that $AP : BP : CP = 1 : 2 : 3$.

(a) Find the size of $\angle APB$.

(b) Give a straight-edge-and-compasses construction of the point P.

9 E Tangents and Radii

Tangents were the object of intensive study in calculus, because the derivative was defined as the gradient of the tangent. Circles, however, were their original context, and the results in the remainder of this chapter are developed without reference to the derivative.

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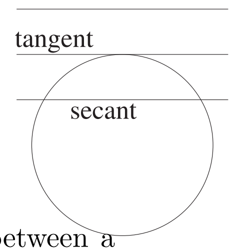
Tangent and Radius: We shall assume that given a circle, any line is a secant crossing a circle at two points, or is a tangent touching it at one point, or misses the circle entirely.



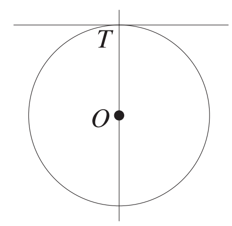
17 DEFINITION: A *tangent* is a line that meets a circle in one point, called the *point of contact*.

We shall also make the following assumption about the relationship between a tangent and the radius at the point of contact.

18 COURSE ASSUMPTION: At every point on a circle, there is one and only one tangent to the circle at that point. This tangent is the line through the point perpendicular to the radius at the point.

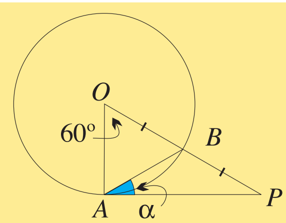
This result can easily be seen informally in two ways. First, a diameter is an axis of symmetry of a circle — this symmetry reflects the perpendicular line at the endpoint T onto itself, and so the perpendicular line cannot meet the circle again, and is therefore a tangent.

Alternatively, if a line ever comes closer than the radius to the centre, then it will cross the circle twice and be a secant, so a tangent at a point T on a circle must be a line whose point of closest approach to the centre is T — but the closest distance to the centre is the perpendicular distance, therefore the tangent is the line perpendicular to the radius.



WORKED EXERCISE: Find α in the diagram below, where O is the centre, and prove that PA is a tangent to the circle.

SOLUTION: $OA = OB$ (radii),
so $\angle OAB = \angle OBA = 60^\circ$ (angle sum of isosceles $\triangle OAB$),
so $BA = OB = PB$ ($\triangle OBA$ is equilateral).
Hence $\alpha = \angle P = 30^\circ$ (exterior angle of isosceles $\triangle BAP$),
so $\angle OAP = 90^\circ$ (adjacent angles).
Hence PA is a tangent to the circle.



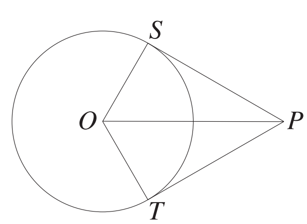
Tangents from an External Point: The first formal theorem about tangents concerns the two tangents to a circle from a point outside the circle.

19 COURSE THEOREM: The two tangents from an external point have equal lengths.

GIVEN: Let PS and PT be two tangents to a circle with centre O from an external point P.

AIM: To prove that $PS = PT$.

CONSTRUCTION: Join OP, OS and OT.



PROOF: In the triangles SOP and TOP:
1. $OS = OT$ (radii),
2. $OP = OP$ (common),
3. $\angle OSP = \angle OTP = 90^\circ$ (radius and tangent),

so $\triangle SOP \equiv \triangle TOP$ (RHS).
Hence $PS = PT$ (matching sides of congruent triangles).

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WORKED EXERCISE: Use the construction established above to prove:

(a) The tangents from an external point subtend equal angles at the centre.

(b) The interval joining the centre and the external point bisects the angle between the tangents.

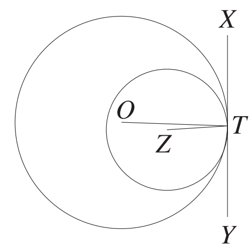
PROOF: Using the congruence $\triangle SOP \equiv \triangle TOP$ established above:

(a) $\angle SOP = \angle TOP$ (matching angles of congruent triangles),

(b) $\angle SPO = \angle TPO$ (matching angles of congruent triangles).

Touching Circles: Two circles are said to *touch* if they have a common tangent at the point of contact. They can touch *externally* or *internally*, as the two diagrams below illustrate.

20 COURSE THEOREM: When two circles touch (internally or externally), the two centres and the point of contact are collinear.



GIVEN: Let two circles with centres O and Z touch at T.

AIM: To prove that O, T and Z are collinear.

CONSTRUCTION: Join OT and ZT,
and construct the common tangent XTY at T.

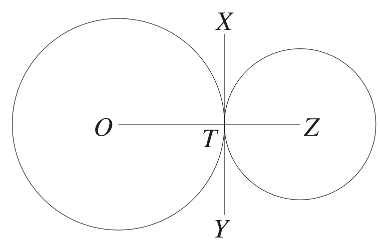
PROOF: There are two possible cases, because the circles can touch internally or externally, but the argument is practically the same in both. Since XY is a tangent and OT and ZT are radii,

$$\angle OTX = 90^\circ \text{ and } \angle ZTX = 90^\circ.$$

Hence $\angle OTZ = 180^\circ$ (when the circles touch externally),

or $\angle OTZ = 0^\circ$ (when the circles touch internally).

In both cases, O, T and Z are collinear.



Direct and Indirect Common Tangents: There are two types of *common tangents* to a given pair of circles:

21 DIRECT AND INDIRECT COMMON TANGENTS: A common tangent to a pair of circles:
- is called *direct*, if both circles are on the same side of the tangent,
- is called *indirect*, if the circles are on opposite sides of the tangent.

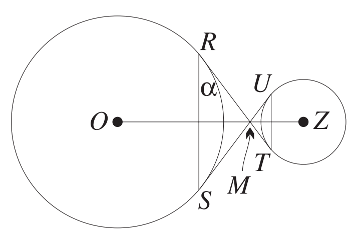
The two types are illustrated in the worked exercise below. Notice that according to this definition, the common tangent at the point of contact of two touching circles is a type of indirect common tangent if they touch externally, and a type of direct common tangent if they touch internally.

WORKED EXERCISE: Given two unequal circles and a pair of direct or indirect common tangents (notice that there are two cases):

(a) Prove that the two tangents have equal lengths.

(b) Prove that the four points of contact form a trapezium.

(c) Prove that their point of intersection is collinear with the two centres.



GIVEN: Let the two circles have centres O and Z.
Let the tangents RT and US meet at M.

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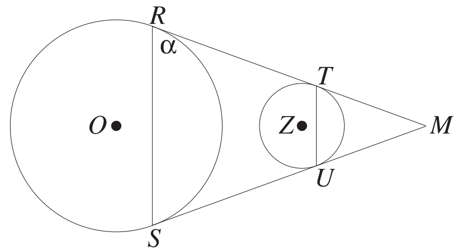


CONSTRUCTION: Join OM and ZM.

AIM: To prove:
(a) $RT = SU$, (b) $RS \parallel UT$,
(c) OMZ is a straight line.

PROOF:
(a) $RM = SM$ (tangents from an external point),
and $TM = UM$ (tangents from an external point),
so $RT = RM - MT = SM - MU = SU$ (direct case),
or $RT = RM + MT = SM + MU = SU$ (indirect case).

(b) Let $\alpha = \angle SRM$.
Then $\angle RSM = \alpha$ (base angles of isosceles $\triangle RMS$),
so $\angle RMS = 180° - 2\alpha$ (angle sum of $\triangle RMS$),
so $\angle TMU = 180° - 2\alpha$ (vertically opposite, or common, angle),
so $\angle UTM = \alpha$ (base angles of isosceles $\triangle TMU$).
Hence $RS \parallel TU$ (alternate or corresponding angles are equal).

(c) From the previous worked exercise, both OM and ZM bisect the angle between the two tangents, and hence

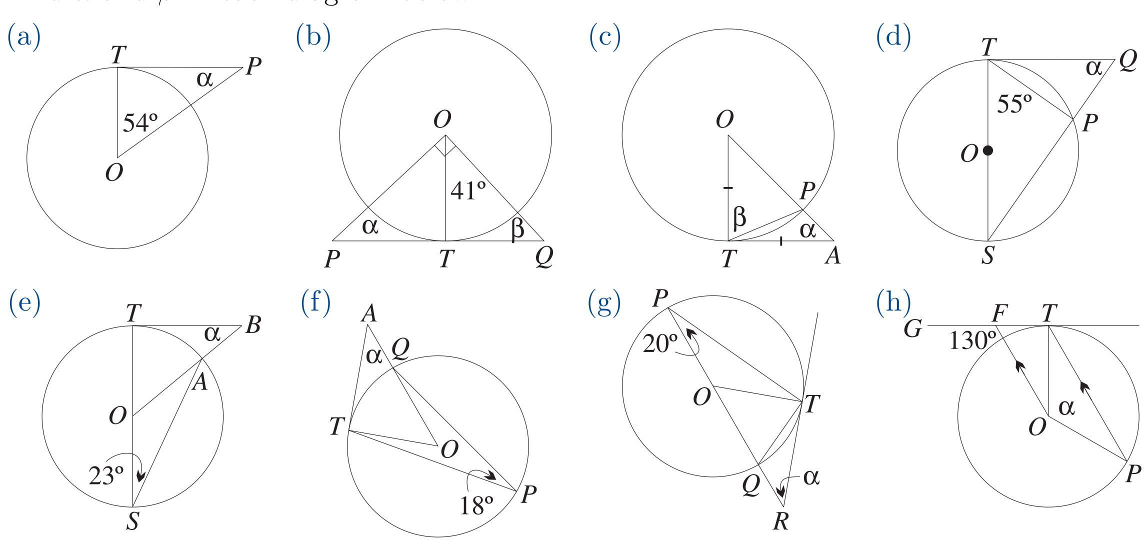
$$\angle RMO = \angle TMZ = 90° - \alpha.$$

In the direct case, OM and ZM must be the same arm of the angle with vertex M. In the indirect case, OMZ is a straight line by the converse of the vertically opposite angles result.

Exercise 9E

NOTE: In each question, all reasons must always be given. Unless otherwise indicated, points labelled O or Z are centres of the appropriate circles, and the obvious lines at points labelled R, S, T and U are tangents.

1. Find α and β in each diagram below.



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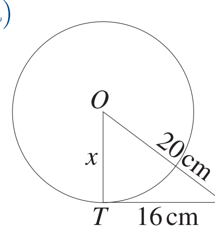
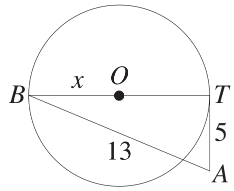
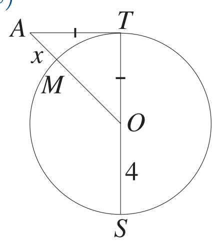
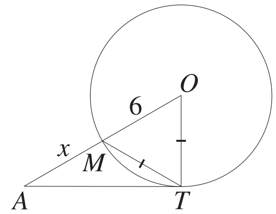
2. Find x in each diagram.

(a)


(b)


(c)


(d)


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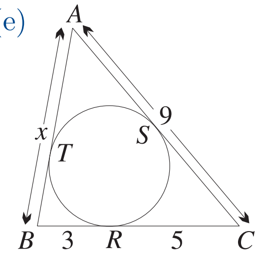

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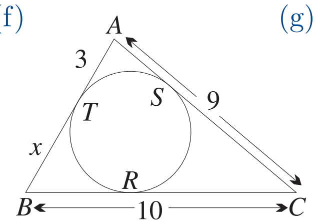
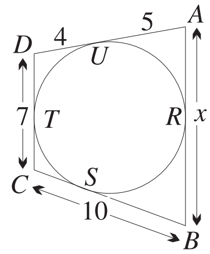
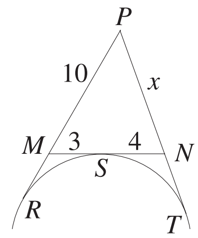
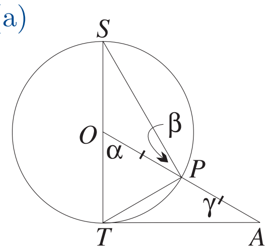
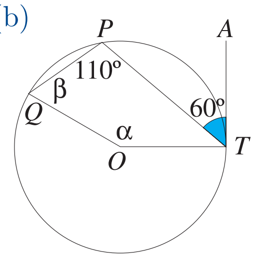


(g)


(h)


3. Find α, β and γ in each diagram below.

(a)


(b)


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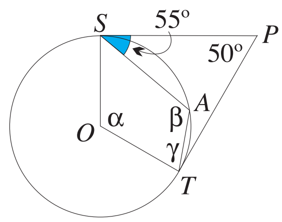

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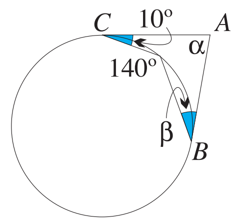
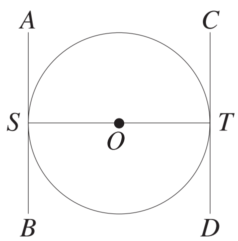
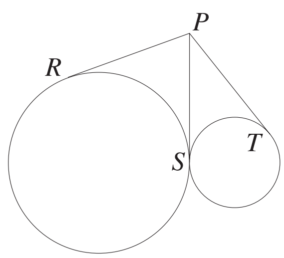
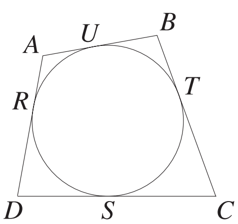


4. (a)


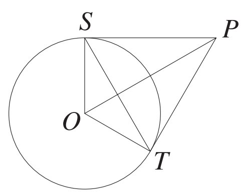
Prove that the tangents at S and at T are parallel.

(b)


Prove that the three tangents PR, PS and PT from the point P on the common tangent at the point S of contact are equal.

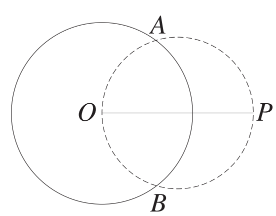
(c)


Prove that $AB + DC = AD + BC$.

(d)


Prove that $OSPT$ is cyclic, and hence that $\angle OST = \angle OPT$ and $\angle TOP = \angle TSP$.

5. CONSTRUCTION: *Construct the tangents to a given circle from a given external point.*
Given a circle with centre O and an external point P, construct the circle with diameter OP, and let the two circles intersect at A and B. Prove that PA and PB are tangents.



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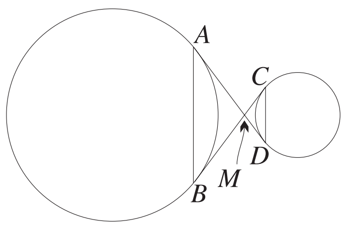
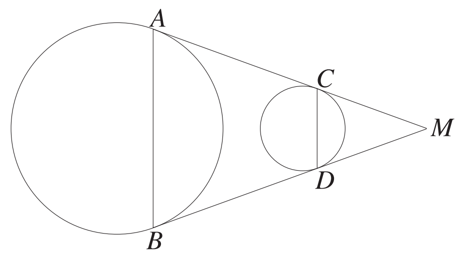
6. (a)



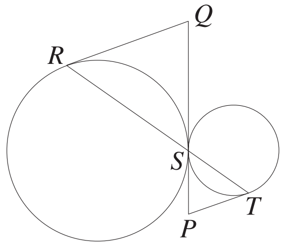
Prove: (i) the indirect common tangents AD and BC are equal, (ii) $AB \parallel CD$.

(b)



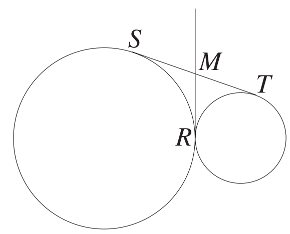
Prove: (i) the direct common tangents AC and BD are equal, (ii) $AB \parallel CD$.

(c)



Prove that $PT \parallel RQ$.

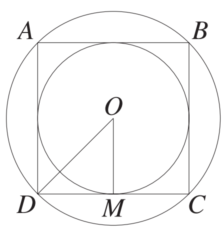
(d)



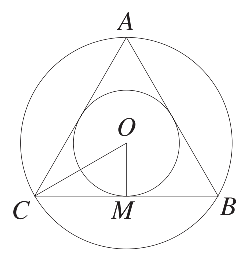
Prove that the common tangent MR at the point of contact bisects the direct common tangent ST, and that $SR \perp TR$.

7. In each diagram, both circles have centre O and the inner circle has radius r. Find the radius of the outer circle if:

(a) $ABCD$ is a square,

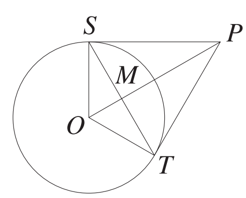


(b) ABC is an equilateral triangle.



DEVELOPMENT

8. (a) THEOREM: *The line joining the centre of a circle to an external point is the perpendicular bisector of the chord joining the points of contact.*
It was proven that $\angle TOM = \angle SOM$ in a worked exercise. Using this, prove by congruence that $TS \perp OP$.



(b) THEOREM: *In the same notation, the* semichord TM *is the geometric mean of the intercepts* PM *and* OM.

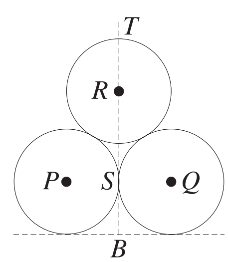
(i) Prove that $\triangle MPT \,|||\, \triangle MTO$.

(ii) Hence prove that $TM^2 = PM \times OM$.

(c) Given that $OM = 7$ and $MP = 28$, find ST.

9. (a) Show that an equilateral triangle of side length $2r$ has altitude of length $r\sqrt{3}$.

(b) Hence find the height of the pile of three circles of equal radius r drawn to the right.



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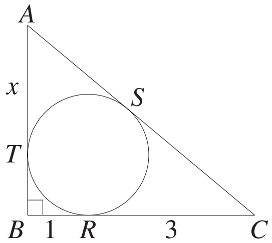
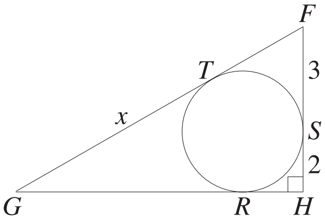
10. In each diagram, use Pythagoras' theorem to form an equation in x, and then solve it. [HINT: In part (c), drop a perpendicular from P to QT.]

(a)



(b)



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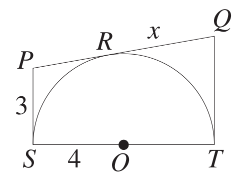

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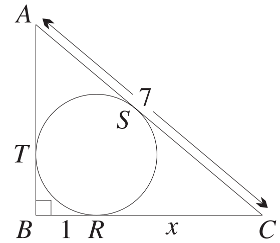
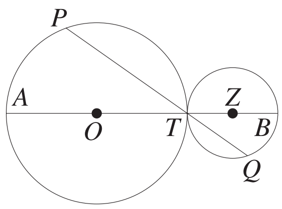
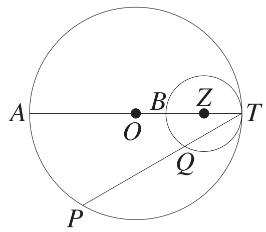


11. In each diagram below, prove: (i) $\triangle PAT \parallel\!| \triangle QBT$, (ii) $AP \parallel QB$.

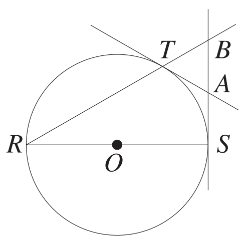
(a)



(b)

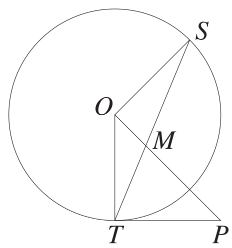


12. (a)



Prove that $BA = AS$.
[HINT: Join TO and TS, then let $\angle R = \theta$.]

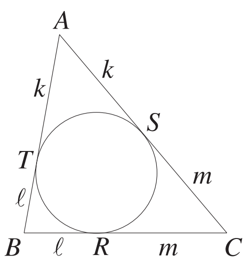
(b)



Given that $PT = PM$, prove that PO is perpendicular to SO.
[HINT: Let $\angle TMP = \theta$.]

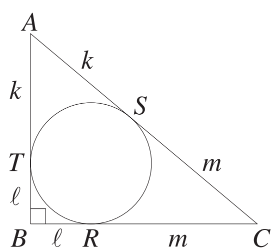
13. CONSTRUCTION: *Construct the circle with a given point as centre and tangential to a given line not passing though the point.* Use the fact that a tangent is perpendicular to the radius at the point of contact to find a ruler-and-compasses construction of the circle.

14. (a)



Suppose that the circle RST is inscribed in $\triangle ABC$. Prove that $k = \frac{1}{2}(-a+b+c)$, $\ell = \frac{1}{2}(a-b+c)$ and $m = \frac{1}{2}(a+b-c)$.

(b)



Suppose further that $\angle ABC = 90^\circ$. Prove that $k = \dfrac{\ell(m+\ell)}{m-\ell}$, and find a, b and c in terms of ℓ and m.

15. THEOREM: *Suppose that two circles touch externally, and fit inside a larger circle which they touch internally. Then the triangle formed by the three centres has perimeter equal to the diameter of the larger circle.* Prove this theorem using a suitable diagram.

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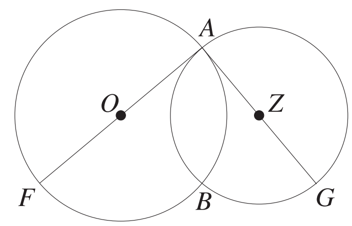
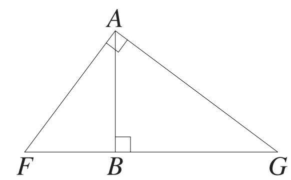
16. (a) Two circles with centres O and Z intersect at A and B so that the diameters AOF and AZG are each tangent to the other circle.



(i) Prove that F, B and G are collinear.

(ii) Prove that $\triangle AGB \parallel\!| \triangle FAB$, and hence prove that AB is the geometric mean of FB and GB (meaning that $AB^2 = FB \times GB$).

(b) Conversely, suppose that AB is the altitude to the hypotenuse of the right triangle AFG. Prove that the circles with diameters AF and AG intersect again at B, and are tangent to AG and AF respectively.



17. (a) Two circles of radii 5 cm and 3 cm touch externally. Find the length of the direct common tangent.

(b) Two circles of radii 17 cm and 10 cm intersect, with a common chord of length 16 cm. Find the length of the direct common tangent.

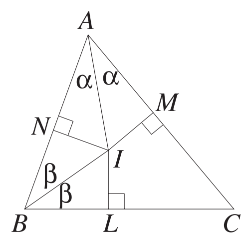
(c) Two circles of radii 5 cm and 4 cm are 3 cm apart at their closest point. Find the lengths of the direct and indirect common tangents.

18. Prove the following general cases of the previous question.

(a) THEOREM: *The direct common tangent of two circles touching externally is the geometric mean of their diameters (meaning that the square of the tangent is the product of the diameters).*

(b) THEOREM: *The difference of the squares of the direct and indirect common tangents of two non-overlapping circles is the product of the two diameters.*

19. THE INCENTRE THEOREM: *The angle bisectors of the vertices of a triangle are concurrent, and their point of intersection (called the* incentre*) is the centre of a circle (called the* incircle*) tangent to all three sides.* In the diagram opposite, the angle bisectors of $\angle A$ and $\angle B$ of $\triangle ABC$ meet at I. The intervals IL, IM and IN are perpendiculars to the sides.



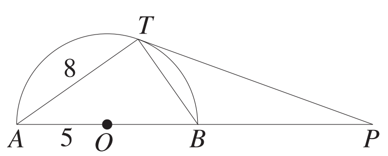
(a) Prove that $\triangle AIN \equiv \triangle AIM$ and $\triangle BIL \equiv \triangle BIN$.

(b) Prove that $IL = IM = IN$.

(c) Prove that $\triangle CIL \equiv \triangle CIM$.

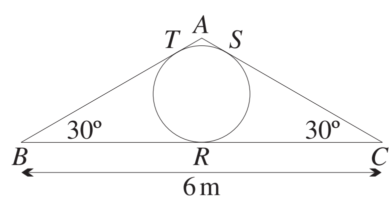
(d) Complete the proof.

20. (a)



TRIGONOMETRY: The figure ATB in the diagram above is a semicircle. Find the exact values of the lengths TP and BP.

(b)



MENSURATION: This window is made in four pieces. Find the area of the small piece AST exactly and approximately.

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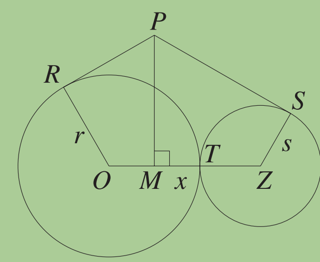
EXTENSION

21. (a) THEOREM: *If two circles touch, the tangents to the two circles from a point outside both of them are equal if and only if the point lies on the common tangent at the point of contact.*
In the diagram to the right, use Pythagoras' theorem to prove that $PR = PS$ if and only if $x = 0$.



(b) THEOREM: *Given three circles such that each pair of circles touches externally, the common tangents at the three points of contacts are equal and concurrent. They meet at the incentre of the triangle formed by the three centres, and the incircle passes through the three points of contact.* Use the result of part (a) to prove this theorem.

22. (a) Three circles of equal radius r are placed so that each is tangent to the other two. Find the area of the region contained between them, and the radius of the largest circle that can be constructed in this region.
(b) Four spheres of equal radius r are placed in a stack so that each touches the other three. Find the height of the stack.

23. (a) CONSTRUCTION: Given two intersecting lines, construct the four circles of a given radius that are tangential to both lines.
(b) CONSTRUCTION: Given two non-intersecting circles, construct their direct and indirect common tangents.

24. (a) THEOREM: Suppose that there are three circles of three different radii such that no circle lies within any other circle. Prove that the three points of intersection of the direct common tangents to each pair of circles are collinear. [HINT: Replace the three circles by three spheres lying on a table, then the direct common tangents to each pair of circles form a cone.]
(b) THEOREM: Prove that the orthocentre of a triangle is the incentre of the triangle formed by the feet of the three altitudes.

9 F The Alternate Segment Theorem

The word 'alternate' means 'the other one'. In the diagram below, the chord AB divides the circle into two segments — the angle $\alpha = \angle BAT$ lies in one of the segments, and the angle APB lies in the other segment. The alternate segment theorem claims that the two angles are equal.

The Alternate Segment Theorem: Stating the theorem verbally:

22 COURSE THEOREM: The angle between a tangent to a circle and a chord at the point of contact is equal to any angle in the alternate segment.

GIVEN: Let AB be a chord of a circle with centre O, and let SAT be the tangent at A. Let $\angle APB$ be an angle in the alternate (other) segment to $\angle BAT$.

AIM: To prove that $\angle APB = \angle BAT$.

CONSTRUCTION: Construct the diameter AOQ from A, and join BQ.

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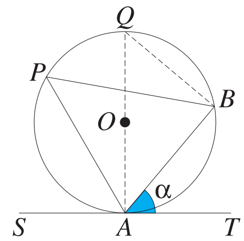


PROOF: Let $\angle BAT = \alpha$.

Since $\angle QAT = 90^\circ$ (radius and tangent),

$\angle BAQ = 90^\circ - \alpha$.

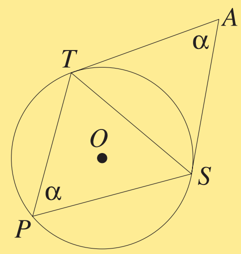
Again, since $\angle QBA = 90^\circ$ (angle in a semicircle),

$\angle Q = \alpha$.

Hence $\angle P = \angle Q = \alpha$ (angles on the same arc BA).

WORKED EXERCISE: In the diagram below, AS and AT are tangents to a circle with centre O, and $\angle A = \angle P = \alpha$.

(a) Find α. (b) Prove that T, A, S and O are concyclic.



SOLUTION:

(a) First, $\angle AST = \alpha$ (alternate segment theorem).

Secondly, $\angle ATS = \alpha$ (alternate segment theorem).

Hence $\triangle ATS$ is equilateral, and $\alpha = 60^\circ$.

(b) $\angle SOT = 120^\circ$ (angles on the same arc ST),

so $\angle A$ and $\angle SOT$ are supplementary.

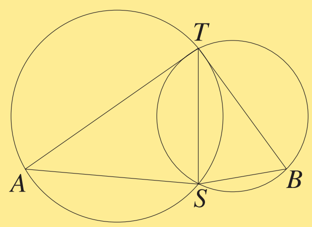
Hence $TASO$ is a cyclic quadrilateral.

WORKED EXERCISE: In the diagram below, AT and BT are tangents.

(a) Prove that $\triangle ATS \;|||\; \triangle TBS$.

(b) Prove that $AS \times BS = ST^2$.

(c) If the points A, S and B are collinear, prove that TA and TB are diameters.



SOLUTION:

(a) In the triangles ATS and TBS:

1. $\angle ATS = \angle TBS$ (alternate segment theorem),
2. $\angle TAS = \angle BTS$ (alternate segment theorem),

so $\triangle ATS \;|||\; \triangle TBS$ (AA).

(b) Using matching sides of similar triangles, $\dfrac{AS}{ST} = \dfrac{ST}{BS}$

$$AS \times BS = ST^2.$$

(c) First, $\angle TSA = \angle TSB$ (matching angles of similar triangles).

Secondly, $\angle TSA$ and $\angle TSB$ are supplementary (angles on a straight line).

Hence $\angle TSA = \angle TSB = 90^\circ$.

Since TA and TB subtend right angles at the circumference, they are diameters.

Exercise 9F

NOTE: In each question, all reasons must always be given. Unless otherwise indicated, points labelled O or Z are centres of the appropriate circles, and the obvious lines at points labelled R, S, T and U are tangents.

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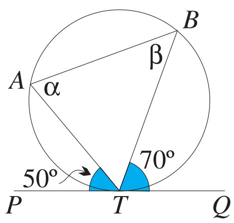
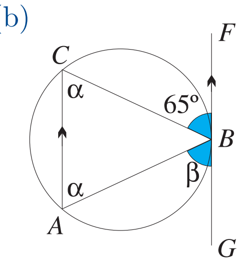
1. State the alternate segment theorem, and draw several diagrams, with tangents and chords in different orientations, to illustrate it.

2. Find α, β, γ and δ in each diagram below.

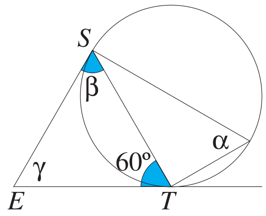
(a)



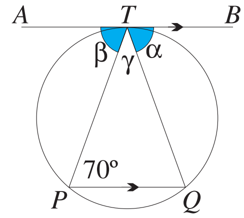
(b)



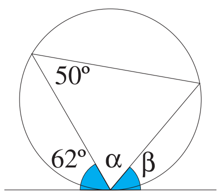
(c)



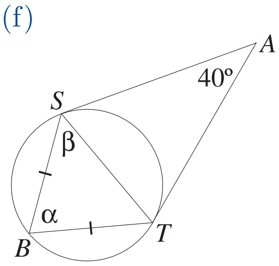
(d)



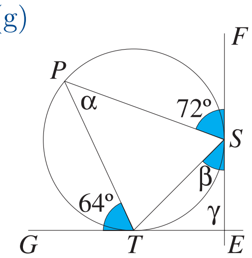
(e)



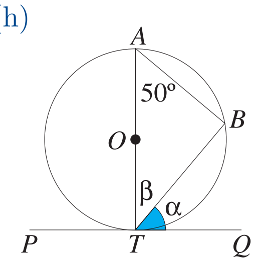
(f)



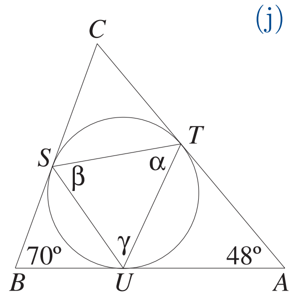
(g)



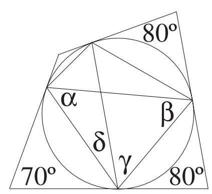
(h)



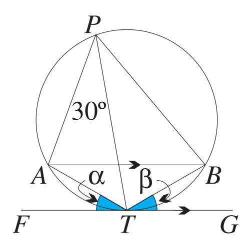
(i)



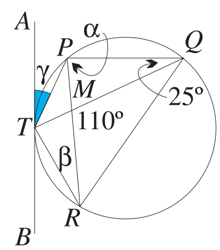
(j)



(k)

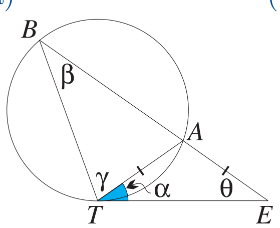


(l)

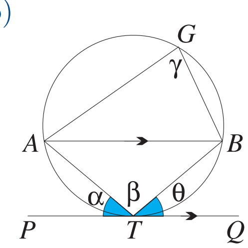


3. In each diagram below, express α, β and γ in terms of θ.

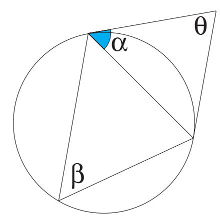
(a)



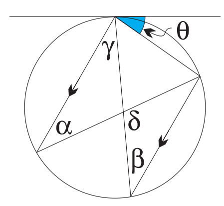
(b)



(c)

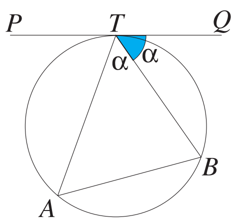


(d)



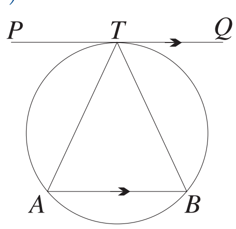
4. In each diagram below, PTQ is a tangent to the circle.

(a)



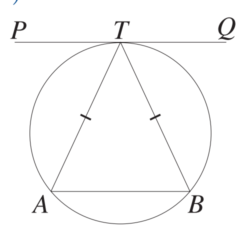
Prove that $BT = BA$.

(b)



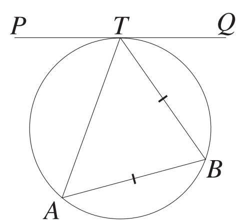
Prove that $TA = TB$.

(c)



Prove that $AB \parallel PTQ$.

(d)



Prove that BT bisects $\angle ATQ$.

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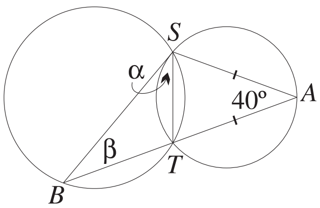
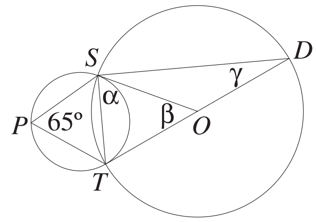
5. (a)



The line SB is a tangent, and $AS = AT$. Find α and β.

(b)



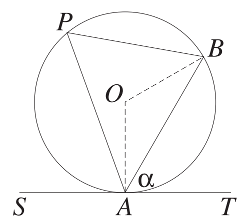
The tangents at S and T meet at the centre O. Find α, β and γ.

DEVELOPMENT

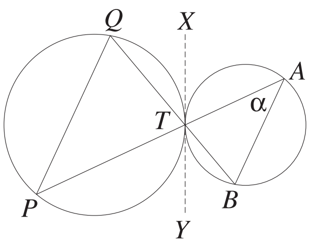
6. ANOTHER PROOF OF THE ALTERNATE SEGMENT THEOREM: Let AB be a chord of a circle, and let SAT be the tangent at A. Let $\angle APB$ be an angle in the alternate segment to $\angle BAT$.

(a) Let $\alpha = \angle BAT$, and find $\angle OAB$.

(b) Find $\angle OBA$ and $\angle AOB$. (c) Hence show that $\angle P = \alpha$.

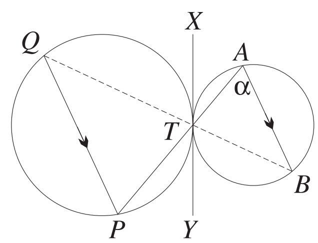


7. (a)



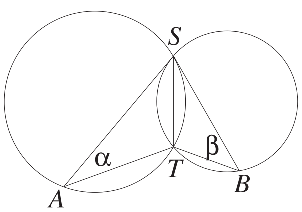
The two circles touch externally at T, and XTY is the common tangent at T. Prove that $AB \parallel QP$.

(b)



The two circles touch externally at T, and XTY is the common tangent there. Prove that the points Q, T and B are collinear.

8. (a)



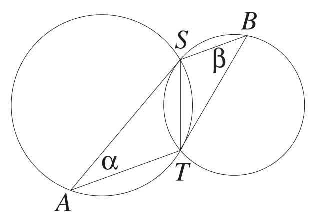
The lines SA and SB are tangents.

(i) Prove that $\angle SAT = \angle BST$.

(ii) Prove that $\triangle SAT \,|||\, \triangle BST$.

(iii) Prove that $AT \times BT = ST^2$.

(b)



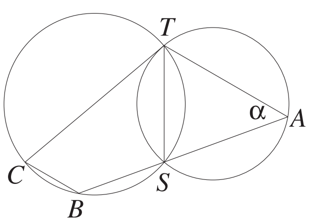
The lines SA and TB are tangents.

(i) Prove that $AT \parallel SB$.

(ii) Prove that $\triangle SAT \,|||\, \triangle BTS$.

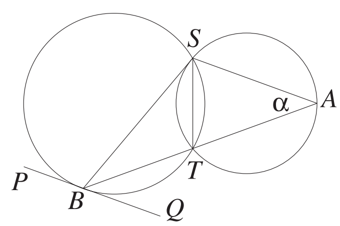
(iii) Prove that $AT \times BS = ST^2$.

9. (a)



The line TC is a tangent. Prove that $TA \parallel CB$.

(b)



The lines SB and PBQ are tangents. Prove that $SA \parallel PBQ$.

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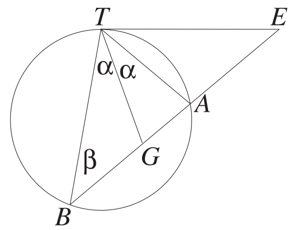
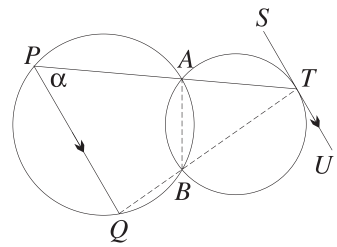
10. (a)



The line TG bisects $\angle BTA$, and ET is a tangent. Prove that $ET = EG$.

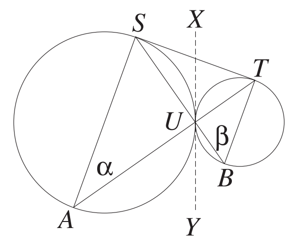
(b)



The line STU is a tangent parallel to PQ. Prove that Q, B and T are collinear.

11. Investigate what happens in question 6, parts (a) and (b), when the two circles touch internally. Draw the appropriate diagrams and prove the corresponding results.

12. ST is a direct common tangent to two circles touching externally at U, and XUY is the common tangent at U.



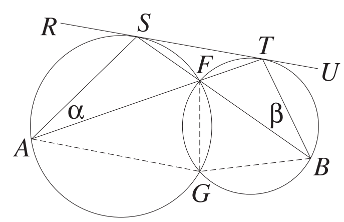
(a) Prove that $AT \perp BS$.

(b) Prove that AS and BT are parallel diameters.

(c) Explain why if the two circles have different diameters, then AB is not a tangent to either circle.

(d) Prove that the circle through S, U and T has centre X and is tangent to AS and BT.

13. $RSTU$ is a direct common tangent to the two circles.



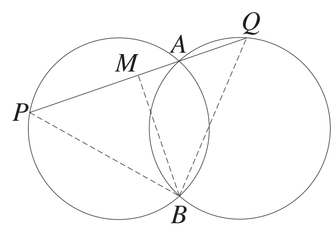
(a) Prove that $\angle RSA = \angle UTB$.

(b) Prove that $\triangle AST \,|||\, \triangle STB$.

(c) Prove that $ST^2 = AS \times BT$.

(d) Prove that if the points A, G and B are collinear, then $\angle SFA = 60°$.

14. A LOCUS PROBLEM: Two circles of equal radii intersect at A and B. A variable line through A meets the two circles again at P and Q.



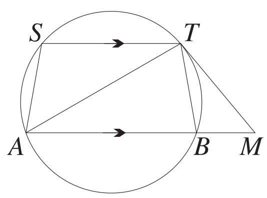
(a) Prove that $\angle QPB = \angle PQB$.

(b) Prove that $BM \perp PQ$, where M is the midpoint of PQ.

(c) What is the locus of M, as the line PAQ varies?

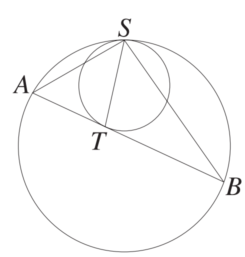
(d) What happens when Q lies on the minor arc AB?

15. (a)



If $ST \parallel AB$ and TM is a tangent, prove that $\triangle TMB \,|||\, \triangle TAS$.

(b)



If the circles are tangent at S, and ATB is a tangent, prove that TS bisects $\angle ASB$.

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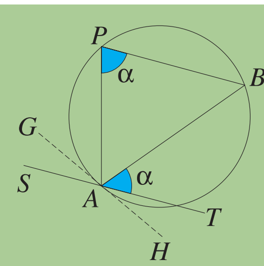
EXTENSION

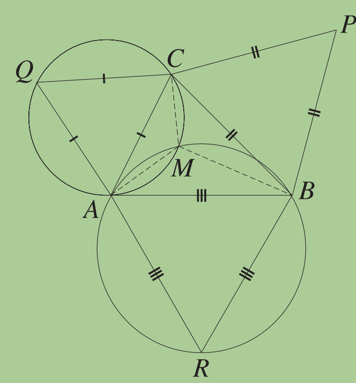
16. THE CONVERSE OF THE ALTERNATE SEGMENT THEOREM: *Suppose that the line SAT passes through the vertex A of $\triangle ABP$ and otherwise lies outside the triangle. Suppose also that $\angle BAT = \angle APB = \alpha$. Then the circle through A, B and P is tangent to SAT.* Construct the circle through A, B and P, and let GAH be the tangent to the circle at A.



(a) Prove that $\angle BAH = \alpha$.

(b) Hence explain why the lines SAT and GAH coincide.

17. THEOREM: *Let equilateral triangles ABR, BCP and CAQ be built on the sides of an acute-angled triangle ABC. Then the three circumcircles of the equilateral triangles intersect in a common point, and this point is the point of intersection of the three concurrent lines AP, BQ and CR.* Construct the circles through A, C and Q and through A, B and R, and let the two circles meet again at M.



(a) Prove that $\angle AMC = \angle AMB = 120°$.

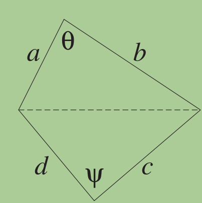
(b) Prove that P, C, M and B are concyclic.

(c) Prove that $\angle AMQ = 60°$.

(d) Hence prove the theorem.

18. The alternate segment theorem has an interesting relationship with the earlier theorem that two angles at the circumference subtended by the same arc are equal. Go back to that theorem (see Box 13), and ask what happens to the diagram as Q moves closer and closer to A. The alternate segment theorem describes what happens when Q is in the limiting position at A.

19. A MAXIMISATION THEOREM: *A cyclic quadrilateral has the maximum area of all quadrilaterals with the same side lengths in the same order.* Let the quadrilateral have fixed side lengths a, b, c and d, and variable opposite angles θ and ψ as shown. Let A be its area.



(a) Explain why $A = \frac{1}{2}ab\sin\theta + \frac{1}{2}cd\sin\psi$.

(b) By equating two expressions for the diagonal, and differentiating implicitly with respect to θ, prove that

$$\frac{d\psi}{d\theta} = \frac{ab\sin\theta}{cd\sin\psi}.$$

(c) Hence prove that $\dfrac{dA}{d\theta} = \dfrac{ab\sin(\theta+\psi)}{2\sin\psi}$, and thus prove the theorem.

9 G Similarity and Circles

The theorems of the previous sections have concerned the equality of angles at the circumference of a circle. In this final section, we shall use these equal angles to prove similarity. The similarity will then allow us to work with intersecting chords, and with secants and tangents from an external point.

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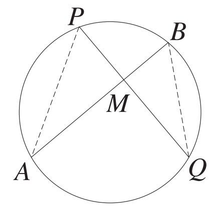
Intercepts on Intersecting Chords: When two chords intersect, each is broken into two intervals called *intercepts*. The first theorem tells us that the product of the intercepts on one chord equals the product of the intercepts on the other chord.

23 COURSE THEOREM: If two chords of a circle intersect, the product of the intercepts on the one chord is equal to the product of the intercepts on the other chord.

GIVEN: Let AB and PQ be chords of a circle intersecting at M.

AIM: To prove that $AM \times MB = PM \times MQ$.

CONSTRUCTION: Join AP and BQ.



PROOF: In the triangles APM and QBM:

1. $\angle A = \angle Q$ (angles on the same arc PB),
2. $\angle AMP = \angle QMB$ (vertically opposite angles),

so $\triangle APM \parallel\!| \triangle QBM$ (AA).

Hence $\dfrac{AM}{QM} = \dfrac{PM}{BM}$ (matching sides of similar triangles),

that is, $AM \times MB = PM \times MQ$.

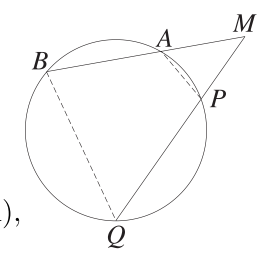
Intercepts on Secants: When two chords need to be produced outside the circle, before they intersect, the same theorem applies, provided that we reinterpret the theorem as a theorem about secants from an external point. The *intercepts* are now the two intervals on the secant from the external point.

24 COURSE THEOREM: Given a circle and two secants from an external point, the product of the two intervals from the point to the circle on the one secant is equal to the product of these two intervals on the other secant.

GIVEN: Let M be a point outside a circle, and let MAB and MPQ be secants to the circle.

AIM: To prove that $AM \times MB = PM \times MQ$.

CONSTRUCTION: Join AP and BQ.



PROOF: In the triangles APM and QBM:

1. $\angle MAP = \angle MQB$ (external angle of cyclic quadrilateral),
2. $\angle AMP = \angle QMB$ (common),

so $\triangle APM \parallel\!| \triangle QBM$ (AA).

Hence $\dfrac{AM}{QM} = \dfrac{PM}{BM}$ (matching sides of similar triangles),

that is, $AM \times MB = PM \times MQ$.

Intercepts on Secants and Tangents: A tangent from an external point can be regarded as a secant meeting the circle in two identical points. With this interpretation, the previous theorem still applies.

25 COURSE THEOREM: Given a circle, and a secant and a tangent from an external point, the product of the two intervals from the point to the circle on the secant is equal to the square of the tangent.
In other words, the tangent is the geometric mean of the intercepts on the secant.

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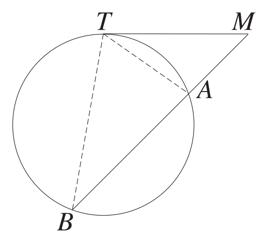
Recall the definitions of arithmetic and geometric means of two numbers a and b:

- The *arithmetic mean* is the number m such that $b - m = m - a$.
  That is, $2m = a + b$ or $m = \dfrac{a+b}{2}$.
- The geometric mean is the number g such that $\dfrac{b}{g} = \dfrac{g}{a}$.
  That is, $g^2 = ab$, or (if g is positive) $g = \sqrt{ab}$.

GIVEN: Let M be a point outside a circle. Let MAB be a secant to the circle, and let MT be a tangent to the circle.

AIM: To prove that $AM \times MB = TM^2$.



CONSTRUCTION: Join AT and BT.

PROOF: In the triangles ATM and TBM:

1. $\angle ATM = \angle TBM$ (alternate segment theorem),
2. $\angle AMT = \angle TMB$ (common),

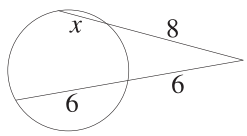
so $\triangle ATM \,|||\, \triangle TBM$ (AA).

Hence $\dfrac{AM}{TM} = \dfrac{TM}{BM}$ (matching sides of similar triangles),

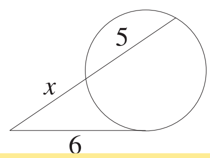
that is, $AM \times MB = TM^2$.

WORKED EXERCISE: Find x in the two diagrams below.

(a)



(b)



SOLUTION:

(a) $8(x + 8) = 6 \times 12$
(intercepts on intersecting chords)
$x + 8 = 9$
$x = 1.$

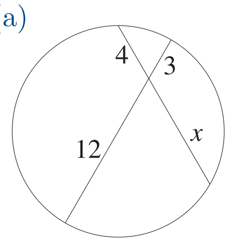
(b) $x(x + 5) = 6^2$ (tangent and secant)
$x^2 + 5x - 36 = 0$
$(x + 9)(x - 4) = 0$
$x = 4$ (x must be positive).

Exercise 9G

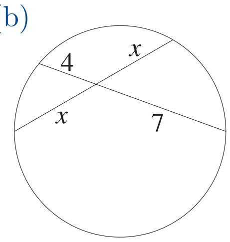
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, points labelled O or Z are centres of the appropriate circles, and the obvious lines at points labelled R, S, T and U are tangents.

1. Find x in each diagram below.

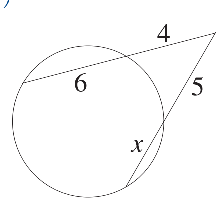
(a)



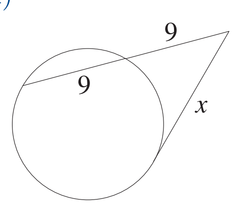
(b)



(c)



(d)



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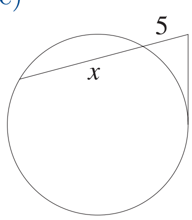
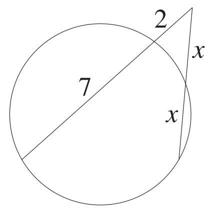
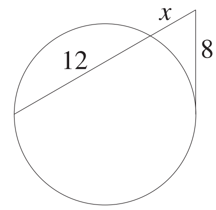
(e)



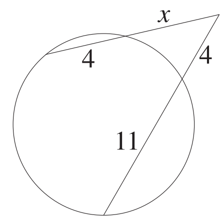
(f)



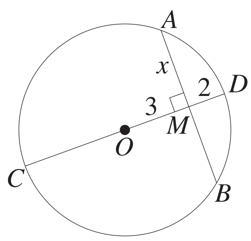
(g)



(h)



2. (a)

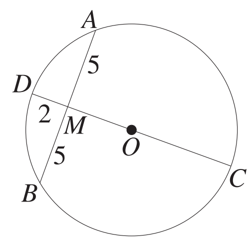


(i) Explain why $MB = x$.

(ii) Find x.

(iii) Find the area of $CADB$.

(b)



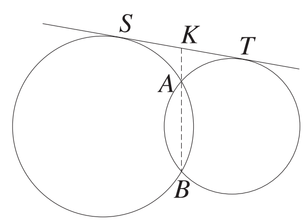
(i) Explain why $CD \perp AB$.

(ii) Find the radius OC.

(iii) Find the area of $CADB$.

DEVELOPMENT

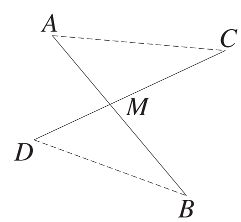
3. THEOREM: *When two circles intersect, the common chord of the two circles bisects each direct common tangent.*
In the diagram, ST is a direct common tangent.



(a) Give a reason why $SK^2 = KA \times KB$.

(b) Hence prove that $SK = TK$.

4. CONVERSE OF THE INTERSECTING CHORDS THEOREM: *If the products of the intercepts on two intersecting intervals are equal, then the four endpoints of the two intervals are concyclic.*
In the diagram opposite, $AM \times BM = CM \times DM$.



(a) Prove that $\dfrac{AM}{CM} = \dfrac{DM}{BM}$.

(b) Prove that $\triangle AMC \,|||\, \triangle DMB$.

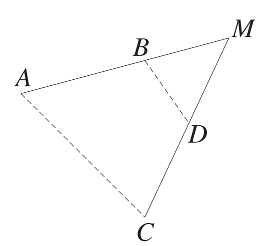
(c) Prove that $\angle CAM = \angle BDM$.

(d) Prove that $ACBD$ is cyclic.

5. CONVERSE OF THE SECANTS FROM AN EXTERNAL POINT THEOREM: *Let two intervals ABM and CDM meet at their common endpoint M, and suppose that*

$$AM \times BM = CM \times DM.$$

Then $ABDC$ is cyclic.



(a) Prove that $\triangle AMC \,|||\, \triangle DMB$.

(b) Prove that $\angle CAM = \angle BDM$.

(c) Prove that $ACBD$ is cyclic.

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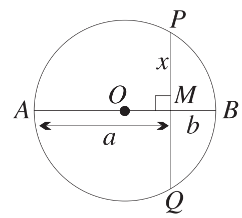
6. THE ARITHMETIC AND GEOMETRIC MEANS:



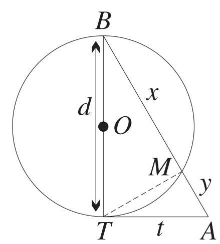
(a) Give a reason why $MQ = x$.

(b) Prove that x is the geometric mean of a and b, that is, $x = \sqrt{ab}$.

(c) Prove that the radius of the circle is the arithmetic mean of a and b, that is, $r = \frac{1}{2}(a + b)$.

(d) Prove that, provided $a \neq b$, the arithmetic mean of a and b is greater than their geometric mean.

7. THE ALTITUDE TO THE HYPOTENUSE:
In the diagram opposite, AT is a tangent.



(a) Show that $t^2 = y(x + y)$.

(b) Show that $d^2 + t^2 - x^2 - y^2 = 2xy$.

(c) Show that $TM \perp BA$.

(d) Where does the circle with diameter TA meet the first circle?

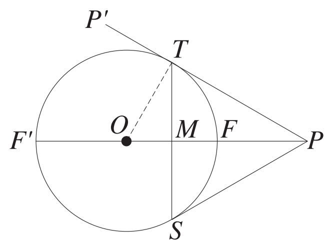
(e) Where does the circle with diameter AB meet the first circle?

(f) Use part (d) to show that $d^2 = x(x + y)$.

(g) Show that $\triangle ATM \;|||\; \triangle TBM \;|||\; \triangle ABT$.

(h) Show that $TM^2 = xy$. (i) Show that $tx = d \times TM$.

8. A CONSTRUCTION OF THE GEOMETRIC MEAN: In the diagram to the right, PT and PS are tangents from an external point P to a circle with centre O.



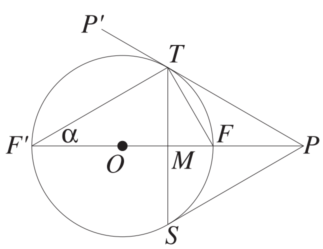
(a) Prove that $\triangle OTM \;|||\; \triangle TPM$.

(b) Prove that $TM^2 = OM \times PM$.

(c) Prove that $OM \times OP = OM \times MP + OM^2$.

(d) Show that OF is the geometric mean of OM and OP (that is, prove that $OF^2 = OM \times OP$).

9. (a)



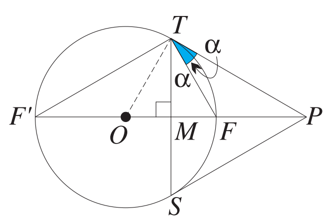
THEOREM: In the diagram, PTP' and PS are tangents. Let $\angle TF'M = \alpha$.

(i) Prove that $\angle FTP = \alpha$.

(ii) Prove that FT bisects $\angle MTP$.

(iii) Prove that $F'T$ bisects $\angle MTP'$.

(b)



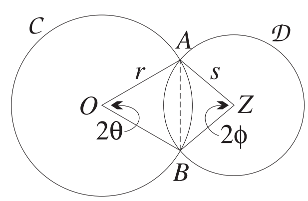
CONVERSE THEOREM: In the diagram, $TS \perp F'OF$, and $\angle FTM = \angle FTP = \alpha$.

(i) Prove that $\angle TF'M = \alpha$.

(ii) Prove that $OT \perp TP$.

(iii) Prove that TP is a tangent.

10. (a) TRIGONOMETRY WITH OVERLAPPING CIRCLES: Suppose that two circles $\mathcal{C}$ and $\mathcal{D}$ of radii r and s respectively overlap, with the common chord subtending angles of 2θ and 2ϕ at the respective centres of $\mathcal{C}$ and $\mathcal{D}$. Show that the ratio $\sin\theta : \sin\phi$ is independent of the amount of overlap, being equal to $s : r$. What happens when 2θ is reflex?



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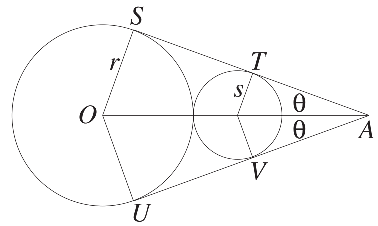
(b) TRIGONOMETRY WITH TOUCHING CIRCLES: Suppose that two circles touch externally, and that their radii are r and s, with $r > s$. Let their direct common tangents meet at an angle 2θ. Show that

$$\frac{r}{s} = \frac{1+\sin\theta}{1-\sin\theta}.$$



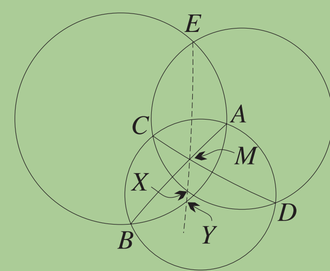
EXTENSION

11. THEOREM: *Given three circles such that each pair of circles overlap, then the three common chords are concurrent.*
In the diagram opposite, the common chords AB and CD meet at M, and the line EM meets the two circles again at X and Y.

(a) By applying the intersecting chord theorem three times, prove that $EM \times MX = EM \times MY$.

(b) Explain why EM must be the third common chord.

(c) Repeat the construction and proof when the common chords AB and CD meet outside the two circles.

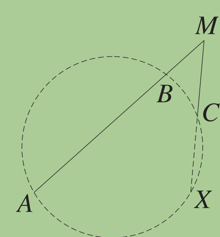


12. THEOREM: *Given three circles such that each pair of circles touch externally, then the three common tangents at the points of contact are concurrent.*
Prove this theorem by making suitable adaptions to the previous proof.

13. CONVERSE OF THE SECANT AND TANGENT THEOREM:
Let the intervals ABM and CM meet at their common endpoint M, and suppose that $MC^2 = MA \times MB$. Then MC is tangent to the circle ABC.
Construct the circle ABC, and suppose by way of contradiction that it meets MC again at X.

(a) Prove that $MA \times MB = MC \times MX$.

(b) Hence prove that C and X coincide.



14. HARMONIC CONJUGATES: In the configuration of question 9:

(a) Prove that M divides $F'F$ internally in the same ratio that P divides $F'F$ externally. (M is called the *harmonic conjugate of* P with respect to F' and F).

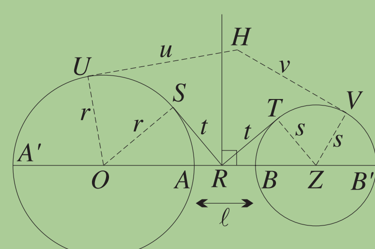
(b) Prove that $F'F$ is the *harmonic mean* of $F'M$ and $F'P$ (meaning that $\dfrac{1}{F'F}$ is the arithmetic mean of $\dfrac{1}{F'M}$ and $\dfrac{1}{F'P}$).

15. THE RADICAL AXIS THEOREM:

(a) Suppose that two circles with centres O and Z and radii r and s do not overlap. Let the line OZ meet the circles at A', A, B and B' as shown, and let $AB = \ell$. Choose R on AB so that the tangents RS and RT to the two circles have equal length t.

(i) Prove that a point H outside both circles lies on the perpendicular to OZ through R if and only if the tangents from H to the two circles are equal.

(ii) Prove that $AR : RB = AB' : A'B$.



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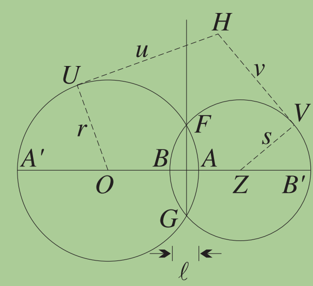
(b) Suppose that two circles with centres O and Z and radii r and s overlap, meeting at F and G. Let the line OZ meet the circles at A', A, B and B' as shown, with $AB = \ell$. Let OZ meet FG at R.



(i) Prove that if H is any point outside both circles, then H lies on FG produced if and only if the tangents from H to the two circles are equal.

(ii) Prove that $AR : RB = AB' : A'B$.

16. CONSTRUCTIONS TO SQUARE A RECTANGLE, A TRIANGLE AND A POLYGON:

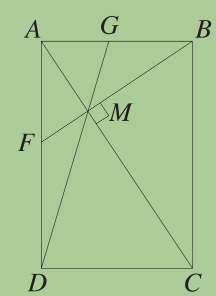
(a) Use the configuration in question 6 to construct a square whose area is equal to the area of a given rectangle.

(b) Construct a square whose area is equal to the area of a given triangle.

(c) Construct a square whose area is equal to the area of a given polygon.

17. CONSTRUCTION: Construct the circle(s) tangent to a given line and passing through two given points not both on the line.

18. GEOMETRIC SEQUENCES IN GEOMETRY: In the diagram below, $ABCD$ is a rectangle with $AB : BC = 1 : r$. The line through B perpendicular to the diagonal AC meets AC at M and meets the side AD at F. The line DM meets the side AB at G.



(a) Write down five other triangles similar to $\triangle AMF$.

(b) Show that the lengths FA, AB and BC form a GP.

(c) Find the ratio $AG : GB$ in terms of r, and find r if AG and GB have equal lengths.

(d) Is it possible to choose the ratio r so that DG is a common tangent to the circles with diameters AF and BC respectively?

(e) Is it possible to choose the ratio r so that the points D, F, G and B are concyclic and distinct?

19. A DIFFICULT THEOREM: Prove that the tangents at opposite vertices of a cyclic quadrilateral intersect on the secant through the other two vertices if and only if the two products of opposite sides of the cyclic quadrilateral are equal.

GO Online Multiple Choice Quiz

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