VISUALIZATION OF KALMAN STEPS

EUGENE YABLONSKI

The applet visualizes the first two steps of a simple Kalman filter. We follow the notations and terminology from the Welch & Bishop's paper [1].

1. Estimating a random constant

Let x and v_1, v_2 be independent *centered* random variables with finite 2nd moments:

$$P_0 = Ex^2;$$
$$R = E(v_1)^2 = E(v_2)^2.$$

Let $z_1 = x + v_1$ and $z_2 = x + v_2$.

We interpret z_1 and z_2 as noisy measurements of x.

Problem. Given z_1 and z_2, find the best linear estimate $\hat{x} = c_1 z_1 + c_2 z_2$ of x in the sense that $E(x - \hat{x})^2$ is as small as possible.

Moreover, perform this task recursively:

1. First, approximate x by a multiple of z_1; that would produce an estimate $\hat{x_1} = K z_1$ for some K;
2. Then, determine $\hat{x}$ as a linear combination of $\hat{x_1}$ and z_2.

2. Geometric Interpretation of the problem

- The space $Span(x, z_1, z_2) = \{c_0 x + c_1 z_1 + c_2 z_2 | c_i \in \mathbb{R}\}$ with the scalar product $u \cdot v = E(uv)$ is a Euclidean space.
- Consider the norm $\|u\| = \sqrt{u \cdot u}$. In our notations:

$$\|x\|^2 = P_0;$$
$$\|v_1\|^2 = \|v_2\|^2 = R.$$

Problem. Find the orthogonal projection $\hat{x_2}$ of the variable x onto the subspace $Span(z_1, z_2) = \{c_1 z_1 + c_2 z_2 | c_i \in \mathbb{R}\}$.

Specifically, implement this as a recursive procedure:

1. First find the projection $\widehat{x_1}$ of x onto $Span(z_1)$;
2. Then, compute $\widehat{x_2}$ as a linear combination of $\hat{x_1}$ and z_2.

3. Kalman estimates

- If $\widehat{x_1} = K_1 z_1$ is the orthogonal projection of x onto the vector z_1, then

$$K_1 = \frac{x \cdot z_1}{||z_1||^2} = \frac{x \cdot (x + v_1)}{||x + v_1||^2} = \frac{||x||^2 + x \cdot v_1}{||x||^2 + ||v_1||^2} = \frac{P_0}{P_0 + R}$$

  since x and v_1 are orthogonal.
- The variance of $\widehat{x_1}$ is

$$||\widehat{x_1}||^2 = ||K_1 z_1||^2 = (K_1)^2 (P_0 + R)^2 = K_1 P_0.$$

- Now, what Welch and Bishop[1] denote by P_1 is

$$\begin{aligned} ||x - \widehat{x_1}||^2 &= ||x||^2 - ||\widehat{x_1}||^2 \\ &= P_0 - K_1 P_0 \\ &= (1 - K_1) P_0 \end{aligned}$$

  by the Pythagorean theorem.

- $z_2 - \widehat{x_1}$ and z_1 are orthogonal since their dot product is

$$\begin{aligned}(z_2 - \widehat{x_1}) \cdot z_1 &= (v_2 + x - \widehat{x_1}) \cdot (v_1 + x) \\ &= v_2 \cdot (v_1 + x) + (x - \widehat{x_1}) \cdot v_1 + (x - \widehat{x_1}) \cdot x \\ &= 0 + 0 + 0.\end{aligned}$$

  Thus, $z_2 - \widehat{x_1}$ and z_1 form an orthogonal basis in the plane spanned by z_1 and z_2.
- Therefore, the projection of x on $span(z_1, z_2)$ is the sum of the projections on z_1 and $z_2 - \widehat{x_1}$:

$$\widehat{x_2} = \widehat{x_1} + K_2\left(z_2 - \widehat{x_1}\right),$$

  where

$$\begin{aligned}K_2 &= \frac{x \cdot (z_2 - \widehat{x_1})}{||z_2 - \widehat{x_1}||^2} \\ &= \frac{(x - \widehat{x_1}) \cdot (z_2 - \widehat{x_1})}{||x - \widehat{x_1} + v_2||^2} \\ &= \frac{||x - \widehat{x_1}||^2}{||x - \widehat{x_1}||^2 + R} \\ &= \frac{P_1}{P_1 + R}\end{aligned}$$

- The variance of $K_2\left(z_2 - \widehat{x_1}\right)$ is

$$\begin{aligned}||K_2(z_2 - \widehat{x_1})||^2 &= \left(K_2\right)^2 ||z_2 - \widehat{x_1}||^2 \\ &= K_2 \frac{P_1}{\cancel{(P_1 + R)}} \cancel{(P_1 + R)}.\end{aligned}$$

- What Welch and Bishop would denote by P_2 is

$$\begin{aligned}||x - \widehat{x_2}||^2 &= ||x - \widehat{x_1}||^2 - ||K_2(z_2 - \widehat{x_1})||^2 \\ &= P_1 - K_2 P_1 \\ &= \left(1 - K_2\right) P_1\end{aligned}$$

  by the Pythagorean theorem. See the illustration.

References

[1] Welch & Bishop, An Introduction to Kalman Filter, TR-95-041

[2] Bernt Oksendal, Stochastic Differential Equations, 5th ed.