Linear Combinations – Activity (Vector Form and Matrix form)

Note: Given two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, a vector $\vec{\mathbf{w}}$ is a **linear combination** of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, if there exists scalars c_1 and c_2 such that

$$c_1\vec{\mathbf{u}} + c_2\vec{\mathbf{v}} = \vec{\mathbf{w}}$$

For example: All vectors $\overrightarrow{\boldsymbol{OP}} = \begin{bmatrix} x \\ y \end{bmatrix}$ can be written as a linear combination of $\overrightarrow{\boldsymbol{e_1}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\overrightarrow{\boldsymbol{e_2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (the standard basis vectors in the xy-plane)

In **Vector form** $c_1\overrightarrow{\boldsymbol{e_1}} + c_2\overrightarrow{\boldsymbol{e_2}} = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ Resulting in $\begin{matrix} 1c_1 + 0\,c_2 = x \\ 0c_1 + 1c_2 = y \end{matrix}$ (By Scalar Multiplication and Vector Addition)

In **Matrix form,** we have $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ which means $[1 \quad 0]\bullet\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = x$ and $[0 \quad 1]\bullet\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = y$

Resulting in $\begin{matrix} 1c_1 + 0\,c_2 = x \\ 0c_1 + 1c_2 = y \end{matrix}$ (By Dot Product)

Note: both forms have the same meaning.

For example, letting $\overrightarrow{\boldsymbol{OP}} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ the *vector form* is $c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

the *matrix form* is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

From either form we get the system of linear equations: $\begin{matrix} 1c_1 + 0\,c_2 = 4 \\ 0c_1 + 1c_2 = 6 \end{matrix}$ and the solution $c_1 = 4,\ c_2 = 6$.

So $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$ is a linear combination of $\overrightarrow{\boldsymbol{e_1}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\overrightarrow{\boldsymbol{e_2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. That is, $\begin{bmatrix} 4 \\ 6 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Note: Every vector and point in the xy-plane can be defined by a linear combination of $\overrightarrow{\boldsymbol{e_1}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\overrightarrow{\boldsymbol{e_2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Since this is true, $\overrightarrow{\boldsymbol{e_1}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\overrightarrow{\boldsymbol{e_2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are called a basis of the xy-plane.

An important aside: Two vectors are linearly independent if one cannot be written as a multiple of the other. Two vectors are dependent if one is a multiple of the other. Two linearly independent vectors define a plane (i.e., *a 2 dimensional vector space*), since ever vector (and consequently every point) in the plane results from the linear combination of these vectors (called basis vectors). A single nonzero vector defines a *1 dimensional vector space*, which is a line.

Consider the basis vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

Can $\begin{bmatrix}4\\6\end{bmatrix}$ be written as a linear combination of vectors $\vec{\mathbf{u}} = \begin{bmatrix}3\\1\end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix}-2\\2\end{bmatrix}$?

You will be required to answer this kind of question in three parts:

- **Algebraic Solution** (Write the vector form of the linear combination; Show how the system of equations is obtained from the vector form; Solve the system of equations, if possible; Using the solution, if it exists, write the vector as a linear combination of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Explain results.)
- **Geometric Solution** (Illustrate graphically, using vectors and their properties, the meaning of the algebraic steps arriving at the solution, where all vectors are drawn and labeled on a grid. Explain results.)
- **Verify Results** (Use the GeoGebra Applet **Linear Combinations.ggb** obtained from GeoGebra Book at https://www.geogebra.org/m/XnfUWvvp Paste the GeoGebra Graphics Window with the vector and the algebraic representations of the solution.)

The following illustrates the three part solution process.

Again consider: Can $\begin{bmatrix}4\\6\end{bmatrix}$ be written as a linear combination of vectors $\vec{\mathbf{u}} = \begin{bmatrix}3\\1\end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix}-2\\2\end{bmatrix}$?

Algebraic Solution (Write the vector form of the linear combination; Show how the system of equations is obtained from the vector form; Solve the system of equations, if possible; Using the solution, write the vector as a linear combination of $\vec{u}$ and $\vec{v}$ if this is possible. Explain results.)

1 $c_1\begin{bmatrix}3\\1\end{bmatrix} + c_2\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix}$

2 $\begin{bmatrix}3c_1\\1c_1\end{bmatrix} + \begin{bmatrix}-2c_2\\2c_2\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix}$

$\begin{bmatrix}3c_1 - 2c_2\\c_1 + 2c_2\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix}$

$\Rightarrow$ $3c_1 - 2c_2 = 4$, $c_1 + 2c_2 = 6$

3 $3c_1 - 2c_2 = 4$
$+\; c_1 + 2c_2 = 6$ } Elimination Method
$4c_1 + 0 = 10$

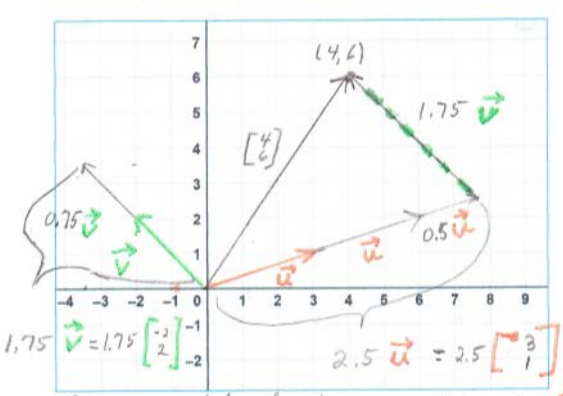
4 $4c_1 = 10$
$\boxed{c_1 = \frac{10}{4} = 2.5}$

$2.5 + 2c_2 = 6$
$2c_2 = 6 - 2.5$
$2c_2 = 3.5$
$\boxed{c_2 = 1.75}$

5 $2.5\vec{u} + 1.75\vec{v} = \begin{bmatrix}4\\6\end{bmatrix}$

6 means the sum of the vectors $2.5\vec{u}$ and $1.75\vec{v}$ is the vector $\begin{bmatrix}4\\6\end{bmatrix}$

Geometric Solution (Illustrate graphically, using vectors and their properties, the meaning of the algebraic steps arriving at the solution, where all vectors are drawn and labeled on a grid. Explain results.)



Note, vectors $\vec{u} = \begin{bmatrix}3\\1\end{bmatrix}$ $\vec{v} = \begin{bmatrix}-2\\2\end{bmatrix}$ on the graph

$2.5\vec{u} = \vec{u} + \vec{u} + 0.5\vec{u}$
(The adding of 3 vectors)

$1.75\vec{v} = \vec{v} + 0.75\vec{v}$
(The adding of 2 vectors)

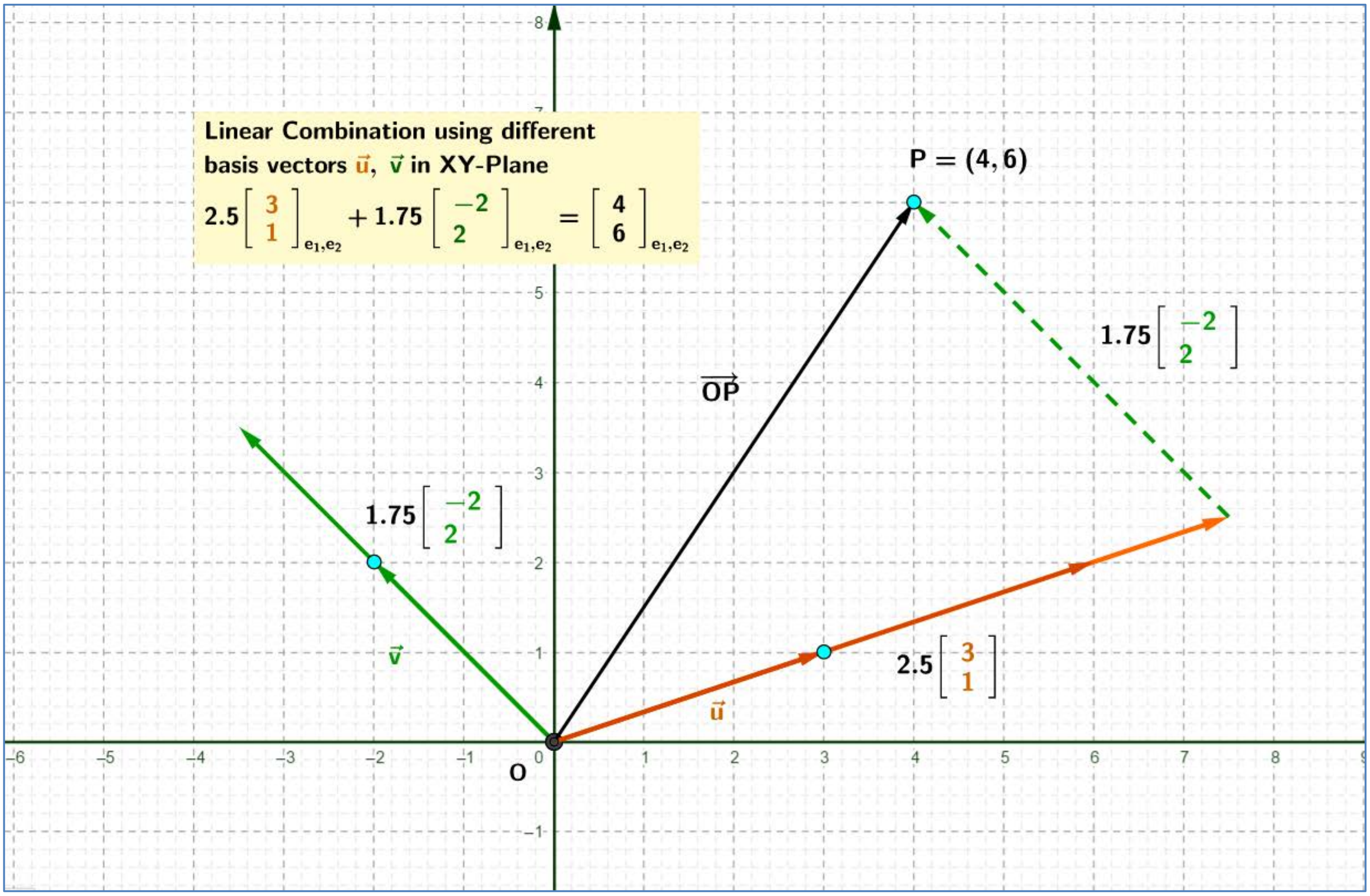
Now, add the two vectors $2.5\vec{u}$ and $1.75\vec{v}$

$2.5\begin{bmatrix}3\\1\end{bmatrix} + 1.75\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}3(2.5)\\1(2.5)\end{bmatrix} + \begin{bmatrix}-2(1.75)\\2(1.75)\end{bmatrix} = \begin{bmatrix}7.5\\2.5\end{bmatrix} + \begin{bmatrix}-3.5\\3.5\end{bmatrix}$

$= \begin{bmatrix}7.5 - 3.5\\2.5 + 3.5\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix}$

Verify Results (Use the GeoGebra Applet **Linear Combinations.ggb** in the GeoGebra Book. Paste the GeoGebra Graphics Window with the vector and the algebraic representations of the solution.)

Note: Any blue terminal point of a vector can be dragged to a new location, creating a new vector.



Can every point in the xy-plane be written as a linear combination of $\vec{u} = \begin{bmatrix}3\\1\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}-2\\2\end{bmatrix}$?

Yes

Explain in terms of these vectors why your answer is correct.

These 2 vectors are linearly independent.

If every point in the xy-plane can be written as a linear combination of $\vec{u} = \begin{bmatrix}3\\1\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}-2\\2\end{bmatrix}$ what is the dimension of the space they define?

Every 2 dimensional space is defined by 2 linearly independent vectors.

1. Determine if $\begin{bmatrix}4\\6\end{bmatrix}$ can be written as a linear combination of $\vec{\mathbf{u}} = \begin{bmatrix}3\\1\end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix}-3\\4\end{bmatrix}$

Algebraic Solution (Write the vector form of the linear combination; Show how the system of equations is obtained from the vector form; Solve the system of equations, if possible; Using the solution, write the vector as a linear combination of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ if this is possible**.** Explain results.)

Geometric Solution Geometric Solution (Illustrate graphically, using vectors and their properties, the meaning of the algebraic steps arriving at the solution, where all vectors are drawn and labeled on a grid. Explain results.)

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Verify Results (Use the GeoGebra Applet **Linear Combinations.ggb** in the GeoGebra Book. Paste the GeoGebra Graphics Window with the vector and the algebraic representations of the solution.)

Can every point in the xy-plane be written as a linear combination of $\vec{\mathbf{u}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$?

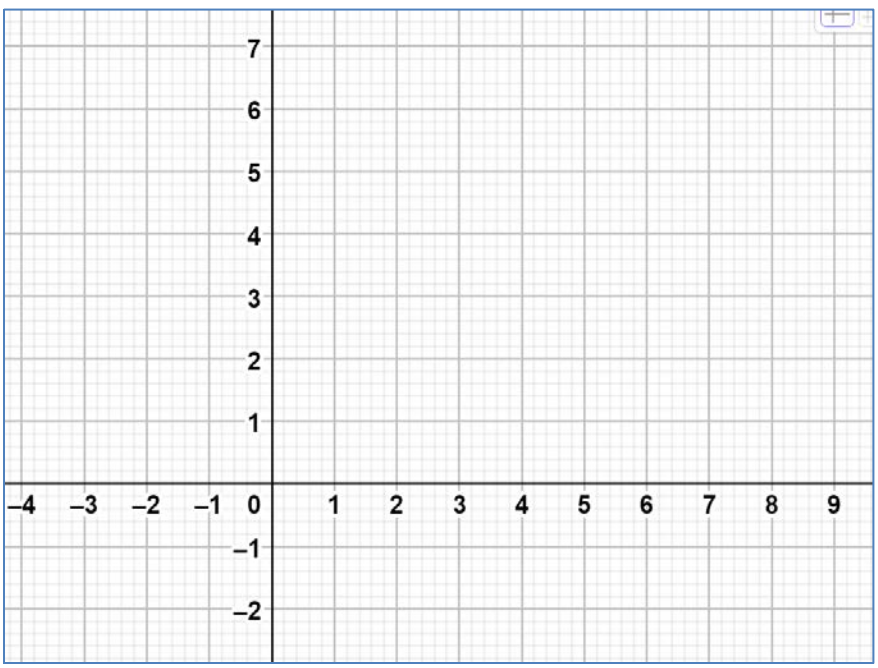
Explain in terms of these vectors why your answer is correct.

If every point in the xy-plane can be written as a linear combination of $\vec{\mathbf{u}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ what is the dimension of the space they define?

2. Determine if $\begin{bmatrix}4\\6\end{bmatrix}$ can be written as a linear combination of the following vectors $\vec{\mathbf{u}} = \begin{bmatrix}2\\1\end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix}-2\\3\end{bmatrix}$

Algebraic Solution (Write the vector form of the linear combination; Show how the system of equations is obtained from the vector form; Solve the system of equations, if possible; Using the solution, write the vector as a linear combination of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ if this is possible**.** Explain results.)

Geometric Solution (Illustrate graphically, using vectors and their properties, the meaning of the algebraic steps arriving at the solution, where all <u>vectors are drawn and labeled</u> on a grid. Explain results.)



Verify Results (Use the GeoGebra Applet **Linear Combinations.ggb** in the GeoGebra Book. Paste the GeoGebra Graphics Window with the vector and the algebraic representations of the solution.)

Can every point in the xy-plane be written as a linear combination of $\vec{\mathbf{u}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$?

Explain in terms of these vectors why your answer is correct.

If every point in the xy-plane can be written as a linear combination of $\vec{\mathbf{u}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ what is the dimension of the space they define?

3. Determine if $\begin{bmatrix}4\\6\end{bmatrix}$ can be written as a linear combination of the following vectors $\vec{\mathbf{u}} = \begin{bmatrix}-4\\-6\end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix}2\\3\end{bmatrix}$

Algebraic Solution (Write the vector form of the linear combination; Show how the system of equations is obtained from the vector form; Solve the system of equations, if possible; Using the solution, write the vector as a linear combination of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ if this is possible**.** Explain results.)

Geometric Solution (Illustrate graphically, using vectors and their properties, the meaning of the algebraic steps arriving at the solution, where all <u>vectors are drawn and labeled</u> on a grid. Explain results.)

Verify Results (Use the GeoGebra Applet **Linear Combinations.ggb** in the GeoGebra Book. Paste the GeoGebra Graphics Window with the vector and the algebraic representations of the solution.)

Can every point in the xy-plane be written as a linear combination of $\vec{\mathbf{u}} = \begin{bmatrix} -4 \\ -6 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$?

Explain in terms of these vectors why your answer is correct.

What is the dimension of the generated space? Explain your answer.

Is $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$ in the vector space defined by the vector $\vec{\mathbf{u}} = \begin{bmatrix} -4 \\ -6 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$?

4. Determine if $\begin{bmatrix}1\\7\end{bmatrix}$ can be written as a linear combination of the following vectors $\vec{u} = \begin{bmatrix}-4\\-6\end{bmatrix}$, $\vec{v} = \begin{bmatrix}2\\3\end{bmatrix}$

Algebraic Solution (Write the vector form of the linear combination; Show how the system of equations is obtained from the vector form; Solve the system of equations, if possible; Using the solution, write the vector as a linear combination of $\vec{u}$ and $\vec{v}$ if this is possible**.** Explain results.)

Geometric Solution (Illustrate graphically, using vectors and their properties, the meaning of the algebraic steps arriving at the solution, where all <u>vectors are drawn and labeled</u> on a grid. Explain results.)

Verify Results (Use the GeoGebra Applet **Linear Combinations.ggb** in the GeoGebra Book. Paste the GeoGebra Graphics Window with the vector and the algebraic representations of the solution.)

Can every point in the xy-plane be written as a linear combination of $\vec{\mathbf{u}} = \begin{bmatrix}-4\\-6\end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix}2\\3\end{bmatrix}$?

Explain in terms of these vectors why your answer is correct.

What is the dimension of the generated space? Explain your answer.

Is $\begin{bmatrix}1\\7\end{bmatrix}$ in the vector space defined by the vectors $\vec{\mathbf{u}} = \begin{bmatrix}-4\\-6\end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix}2\\3\end{bmatrix}$?