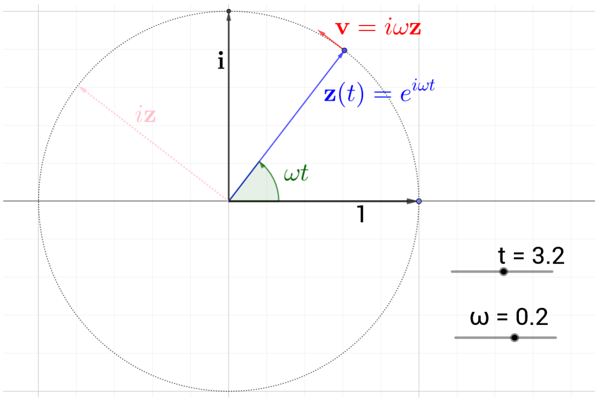
Circular motion and Complex Exponential (Eugene Yablonski)

1. CIRCULAR MOTION WITH CONSTANT SPEED

Exercise 1.1. Your boat makes a circle of radius $r = 1$km at a constant angular speed $\omega = 1°/\text{min} = \frac{\pi}{180}$ rad/min.
If your initial position $\boldsymbol{x}_0$ has polar coordinates $(1, \phi)$, what is your position at time t?

SOLUTION. By time t, you've traveled ωt rad. Let $A(\omega t)$ be the matrix of rotation by angle ωt. Then your new position is

$$\boldsymbol{x}(t) \;=\; A(\omega t)\mathbf{x}_0 \;=\; \begin{bmatrix} \cos(\omega t + \phi) \\ \sin(\omega t + \phi) \end{bmatrix}.$$



Exercise 1.2. Describe the same movement on a complex plane.

ANSWER:

$$\boxed{z(t) \;=\; e^{i\omega t} z_0} \;=\; e^{i(\omega t + \phi)}.$$

Remark. Your **velocity vector** is

$$\boldsymbol{z}'(t) = \omega\, i\boldsymbol{z}(t).$$

Multiplication by i rotates $\boldsymbol{z}$ by 90°, so that the *velocity vector* $\boldsymbol{z}'(t)$ is perpendicular to the current radius-vector $\boldsymbol{z}(t)$.

Your displacement over $\Delta t = 1$ min is

$$\Delta z \;\approx\; \boldsymbol{z}'(t)\Delta t \;=\; \omega\, i\boldsymbol{z}(t)1,$$

that is you move about $\omega = \frac{\pi}{180}$ km in the direction $iz(t)$ tangent to the circle.