FALLING OBJECT, WITH DRAG FORCE

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Introduction

Consider an object falling vertically in air, under the influence of gravity. It experiences a drag force, which opposes the downward motion. This force may be modeled as varying proportionally with the velocity, as

$$F_d = C_d\, A\, v^p$$

where C_d is the drag coefficient, A is the cross-sectional area in the direction of motion, v is the time-dependent velocity, and p is either 1 or 2. The value of 1 applies for slower motion, in a viscous fluid, while p=2 applies for more realistic examples. We want to find solutions for this motion (position, velocity, acceleration as functions of time) for both cases.

The approach we take depends on how we want to use the force-balance (net force) information. We obtain the acceleration of the object using Newton's Second Law, and we know that this acceleration is the second derivative of position, and it also is the first derivative of the velocity. This will give us two choices for a differential equation. *A differential equation is an algebraic relation that expresses the rate of change of a dependent variable with respect to an independent variable.* The latter is, in many physics problems, time.

Many methods are available for solving differential equations, but these are beyond the mathematical level of this course. Solutions will be presented below, using various methods, but they will not be discussed in detail. When we consider our solution options, we have the situation shown in Table 1.

	$a = \dfrac{d^2y}{dt^2}$	$a = \dfrac{dv}{dt}$
position y(t)	solve	integrate
velocity v(t)	differentiate once	solve
acceleration a(t)	differentiate twice	differentiate once

Table 1. Solution options for acceleration problems.

Generally it will be easier to solve a first-order ordinary[1] differential equation (ODE) than a second-order, but with some methods, notably Laplace transforms, it doesn't make much difference. Note that the solution for a first-order ODE requires one initial condition (such as the velocity at time zero), while a second-order ODE requires two initial conditions (usually the velocity and position at time zero). We may or may not know these initial conditions.

Next we proceed to develop the solutions; example plots showing both sets of solutions (for $p=1$ and $p=2$) are presented last.

Linear Dependence (p=1)

First we define the coordinate system as y vertical, positive upward, in the usual sense. The object begins its motion at $y(0) = h$, with zero initial velocity (in either direction). The force balance is

$$F_{net} = m\,a = -m\,g - k\,v$$

where k is $C_d\,A$, and the sign of the second term reflects the fact that the velocity is negative (i.e., toward the ground). From this we have

$$a = \frac{d^2y}{dt^2} = \frac{dv}{dt} = -g - \frac{k}{m}v \qquad (1)$$

As discussed above, from here we have two options. We can solve for the position y, using the second order ODE

$$\frac{d^2y}{dt^2} = -g - \alpha\frac{dy}{dt}$$

with $\alpha = k/m$, having units of s^{-1}, and then differentiate the solution once for the velocity, and again for the acceleration. Or, we can write a first-order ODE in the velocity

$$\frac{dv}{dt} = -g - \alpha\,v$$

and differentiate this solution to get the acceleration, and integrate the solution to find the position.

First-Order ODE Solution

The approaches give the same result, so we begin with the first-order ODE. This is readily solved using any of several elementary methods; a straightforward choice is a convolution integral, so that

[1] Ordinary as opposed to *partial* differential equations, where the dependent variable is a function of more than one independent variable.

$$\boxed{\begin{aligned} v(t) &= \exp\left(-\alpha t\right) \int_0^t -g \exp\left(\alpha \tau\right) d\tau \\ &= \frac{-g}{\alpha}\left[1 - \exp\left(-\alpha t\right) \right] \end{aligned}} \tag{2}$$

The limit of (2) as time increases is the "terminal velocity"

$$\lim_{t\to\infty} \left\{ \frac{-g}{\alpha}\left[1 - \exp\left(-\alpha t\right) \right] \right\} = \frac{-g}{\alpha} = v_T \tag{3}$$

The velocity limit as the drag constant k approaches zero (the usual "vacuum" assumption) can be shown[2] to be

$$\lim_{\alpha\to 0} \left\{ \frac{-g}{\alpha}\left[1 - \exp\left(-\alpha t\right) \right] \right\} = -g\, t$$

as we would expect.

The acceleration is the time derivative of (2), which is

$$\boxed{a(t) = -g \exp\left(-\alpha t\right)} \tag{4}$$

which is just $-g$ at time zero, and which approaches zero as time increases. Indeed, as the acceleration becomes small, we have the "terminal velocity" condition. This condition of small acceleration means that the forces are (nearly) in balance, so we can easily find the terminal velocity without calculus, since in magnitude

$$m\, g = k\, v_T$$

and v_T is the same as (3). Note that this velocity is only reached asymptotically, not exactly.

The position is obtained by integrating (2), which yields

$$\boxed{y(t) = h + \frac{g}{\alpha^2}\left[1 - \exp\left(-\alpha t\right) \right] - \frac{g}{\alpha}\, t} \tag{5}$$

This is a somewhat unusual function in that it blends a linear term and an exponential term. As the latter vanishes with time, the position begins to change linearly. This is sensible, because the velocity has become constant. The time required for this condition (i.e., the time to reach the terminal velocity) is about

$$\varphi = \frac{7}{\alpha}$$

which will cause the factor in brackets to be 0.999. This function (5) has a natural lower bound of zero, when the object hits the ground. We can approximate the time T required to hit the ground, if it is larger than φ, after the exponential has vanished, with

$$T \approx \frac{h\alpha}{g} + \frac{1}{\alpha}$$

This works reasonably well as long as α is not too small. As above we can consider the motion if the drag constant α approaches zero, by taking the limit of (5). This produces

$$\lim_{\alpha\to 0} \left\{ y(t) \right\} = h - \frac{g\, t^2}{2}$$

which again is not especially obvious[3]. *Note that we cannot, in general, just substitute zero for alpha in these solutions, to see what happens in the "no-drag" case.* We must take a limit.[4] However, we can return to the original ODE and use zero for k, and then develop the solutions for that case. This will of course yield the familiar expressions for the position, velocity, and acceleration for free-fall in a vacuum.

We observe that the solutions for the time-dependent acceleration and velocity satisfy the original differential equation (1). If we use (4) and (2) in (1) and simplify, it will be seen that the resulting equation is correct.

Second-Order ODE Solution

The second-order ODE, for the position, can be solved by several methods. A convenient technique which has wide usefulness is Laplace transforms. (The first-order ODE can of course be solved this way, as well.) This will give

$$s^2\, \Im - s\, h = -\frac{g}{s} - \alpha\left(s\, \Im - h \right)$$

where s is the Laplace variable and $\Im$ is the transform of $y(t)$. From this we find

$$\Im = \frac{h\left(s + \alpha \right) - \frac{g}{s}}{s\left(s + \alpha \right)}$$

and the inverse transform $\Im^{1}$ is the same as (5). Differentiating this once gives (2), and twice gives (4).

Quadratic Dependence (p=2)

In this case we have the drag force varying as the square of the time-dependent velocity. This leads to

$$a = \frac{d^2 y}{dt^2} = \frac{dv}{dt} = -g - \beta\, v^2$$

[2] This requires the use of L'Hospital's rule.

[3] This also uses L'Hospital.

[4] Simple substitution will work for (4) but not for (2) or (5).

where β is similar to α but using the drag coefficient that applies in the quadratic-drag situation, and it has units of m^{-1}. (We would not expect this coefficient to be the same as that in the linear-drag case.) Again we can write a first- or second-order ODE:

$$\frac{d^2 y}{dt^2} = -g - \beta\left(\frac{dy}{dt}\right)^2$$
$$\frac{dv}{dt} = -g - \beta v^2$$

In this case it will be simpler to work with the first-order ODE. Note that this is a nonlinear differential equation. We will subscript the results below with a "2" to indicate that the expressions apply for the quadratic-drag case.

Solutions

Using separation of variables, it can be shown[5] that the velocity is

$$\boxed{\begin{aligned} v_2(t) &= \sqrt{\frac{g}{\beta}}\left[\frac{1 - \exp\left(2\sqrt{g\beta}\; t\right)}{1 + \exp\left(2\sqrt{g\beta}\; t\right)}\right] \\ &= -\sqrt{\frac{g}{\beta}}\tanh\left(\sqrt{g\beta}\; t\right)\end{aligned}} \qquad (6)$$

This can be differentiated[6] to give the acceleration

$$\boxed{\begin{aligned} a_2(t) &= \frac{-g}{\cosh^2\left(\sqrt{g\beta}\; t\right)} \\ &= -g\left\{1 - \tanh^2\left(\sqrt{g\beta}\; t\right)\right\}\end{aligned}} \qquad (7)$$

The initial acceleration from (7) is again –*g*, as we would expect, and this acceleration approaches zero (since the *tanh* approaches unity) with increasing time. Finally, we can integrate[7] (6) to get the position

$$\boxed{y_2(t) = h - \frac{1}{\beta}\ln\left[\cosh\left(\sqrt{g\beta}\; t\right)\right]} \qquad (8)$$

As before we can obtain a terminal speed by noting that (6) approaches

$$v_T = -\sqrt{\frac{g}{\beta}}$$

as time increases. Here the time required to closely approach the terminal velocity is

$$\varphi_2 = \frac{4}{\sqrt{g\beta}}$$

since *tanh(4)* is about 0.999. The time *T* required for the object to hit the ground can be obtained by setting (8) equal to zero, which gives

$$T_2 = \frac{1}{\sqrt{g\beta}}\cosh^{-1}\left[\exp\left(h\beta\right)\right]$$

Unlike the linear case (*p=1*), this expression approaches the correct result in the limit of small β, that is, little drag; thus

$$\lim_{\beta\to 0}\frac{1}{\sqrt{g\beta}}\cosh^{-1}\left[\exp\left(h\beta\right)\right] = \sqrt{\frac{2h}{g}}$$

which is just the no-drag case.

For plotting purposes it would be useful to match the drag coefficients so that both cases reach the same terminal velocity. This happens when

$$-\frac{g}{\alpha} = -\sqrt{\frac{g}{\beta}} \;\; ; \;\; \beta = \frac{\alpha^2}{g}$$

With this value for β, the quadratic-drag solutions are

$$y_2(t) = h - \frac{g}{\alpha^2}\ln\left\{\cosh\left(\alpha t\right)\right\}$$

$$v_2(t) = -\frac{g}{\alpha}\tanh\left(\alpha t\right)$$

$$a_2(t) = -g\left[1 - \tanh^2\left(\alpha t\right)\right]$$

It can be shown, by taking limits, that the expressions (6), (7), and (8) will again produce the free-fall-in-a-vacuum results as the drag coefficient, implemented here in β, approaches zero. Thus,

$$\lim_{\beta\to 0}\left\{h - \frac{1}{\beta}\ln\left[\cosh\left(\sqrt{g\beta}\; t\right)\right]\right\} = h - \frac{g t^2}{2}$$

$$\lim_{\beta\to 0}\left\{-\sqrt{\frac{g}{\beta}}\tanh\left[\sqrt{g\beta}\; t\right]\right\} = -g t$$

and the acceleration result of -*g* we can see by inspection.

[5] This solution method is considerably beyond the level of this course. Those interested should see, e.g., P. V. O'Neil, *Advanced Engineering Mathematics*, 3rd Ed., Wadsworth (1991), pp.61-63.

[6] J. J. Tuma, *Engineering Mathematics Handbook*, 3rd Ed., McGraw- Hill (1987), p. 67.

[7] *ibid*, p. 375.

Example Plots

Finally we have four figures, one each for position and acceleration and two for velocity, for a range of drag coefficients, for both the linear and quadratic cases. The quadratic drag is "matched" to the linear case, to give the same terminal velocity. The thick lines are for the no-drag case. The parameters are arbitrarily set to m=10Kg; k=5Kg/s; the resulting $\alpha = k / m$ is multiplied by 1 (top curve in Fig. 1), 0.5, 0.3, 0.1, and 0. Note that the quadratic-case solutions for α=0 match the zero-drag case (thick line) exactly.

In Fig. 2 we see that the time required to attain the terminal velocity varies inversely with α and is shorter for the quadratic drag than the linear. For the matched terminal velocities, we have that

$$\varphi_1 = \frac{7}{\alpha} \; ; \quad \varphi_2 = \frac{4}{\alpha}$$

so that the linear case takes almost twice the time to attain the terminal velocity. Figure 3 shows a zoom into this data, with the predicted times indicated by a triangle (linear) and square (quadratic). We can also see this difference in Fig. 1, where the dotted lines (quadratic drag) attain the zero-acceleration level faster than the solid lines (linear drag).

In Fig. 4, the position endpoint predictions were, in seconds: 53.0, 29.5, 22.0, 25.1, and infinite, for the linear-drag solutions. The correct zero-drag time is 14.3 seconds. For the quadratic-drag solutions the endpoints were 52.4, 28.3, 19.9, 14.9, and 14.3 seconds. The latter agree well with the indicated times on the plot.

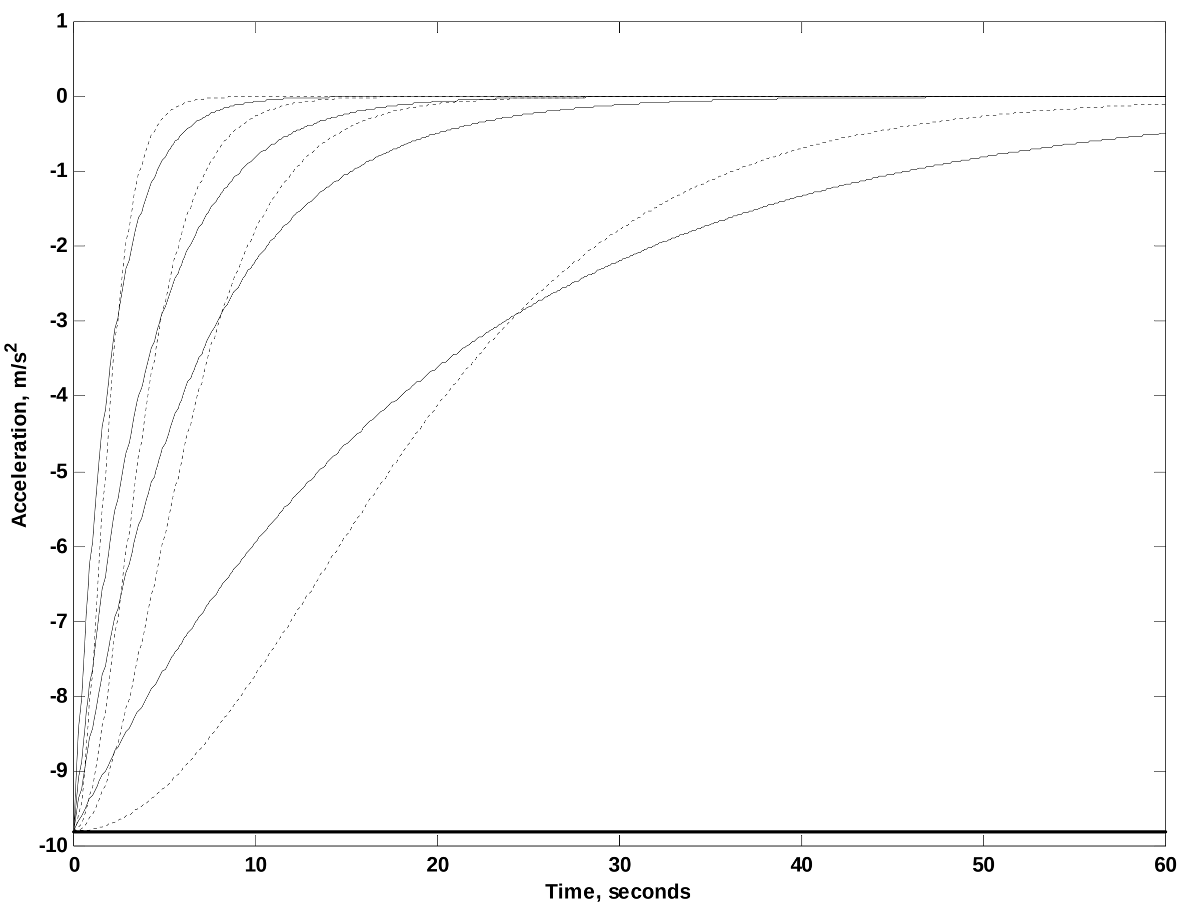

Figure 1. Accelerations vs. time. Solid lines, p=1; dotted lines, p=2.

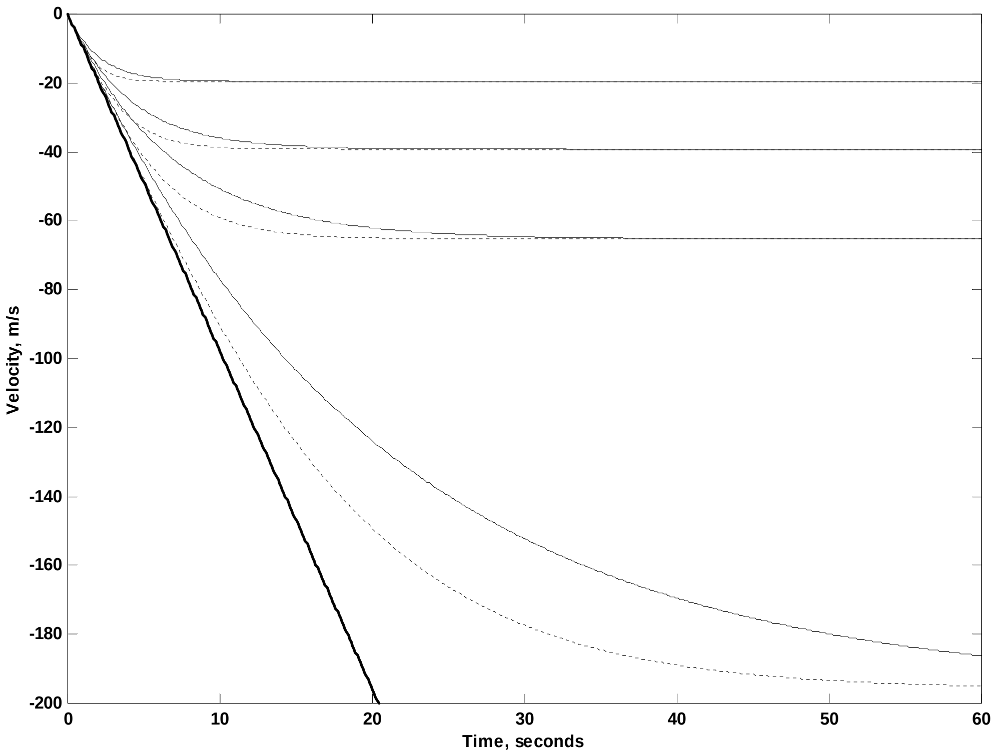

Figure 2. Velocities vs. time.

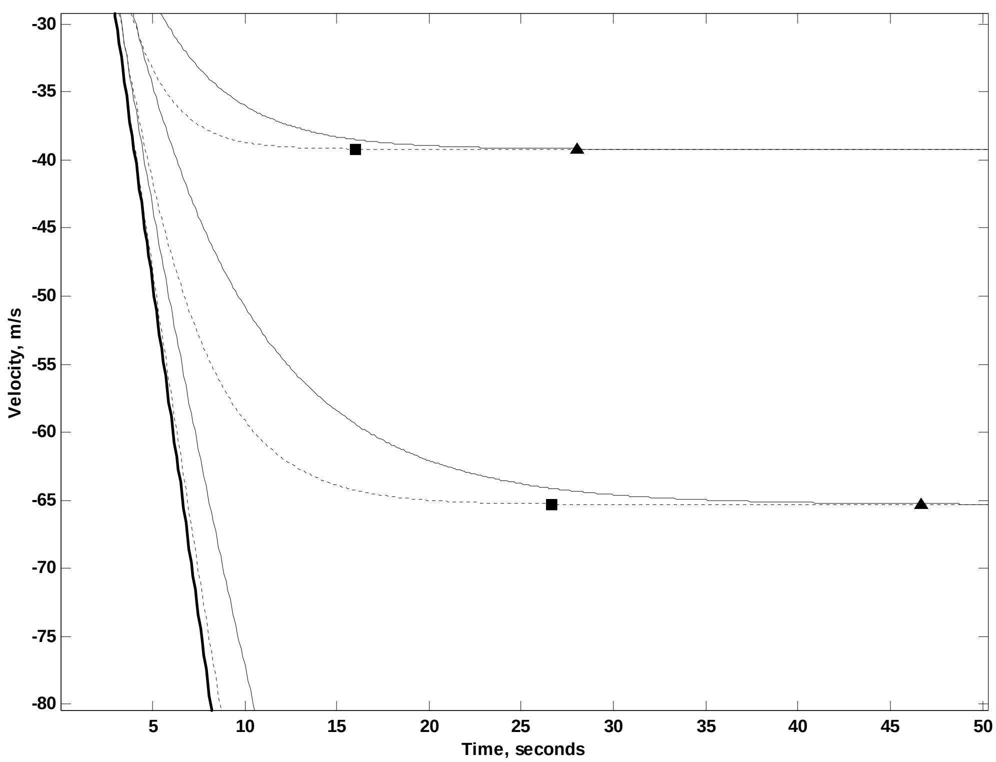

Figure 3. Velocities, showing predicted time-to-terminal-velocity; square=quadratic, triangle=linear.

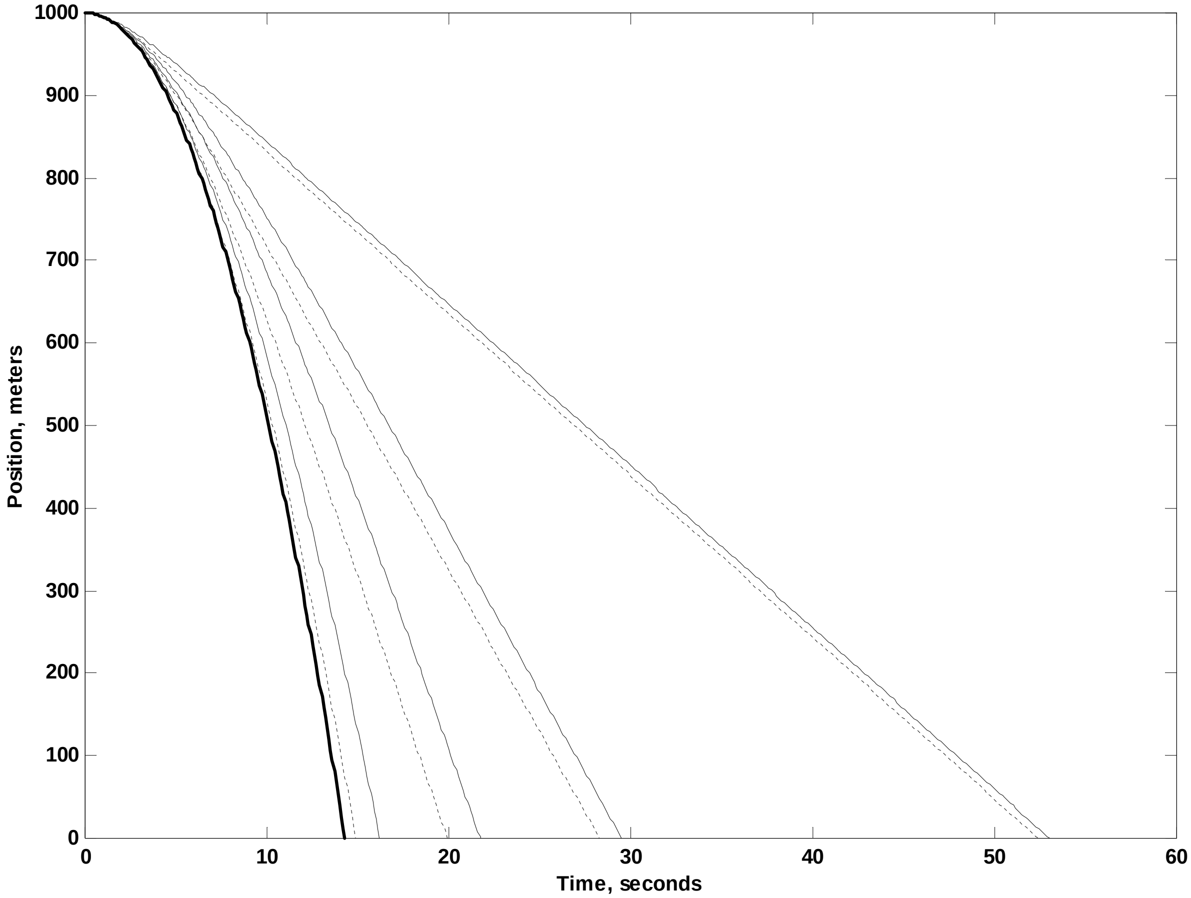

Figure 4. Positions vs. time.