# An application of the program GeoGebra

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**Abstract:** The dynamic geometrical system (DGS) GeoGebra has been introduced and applied for investigations concerning new notable points (called in the following as L: locus points) in 2D and 3D. It is demonstrated how much visual treatment of problems can help in finding analytical solutions. Among examples, L points of plane triangles and tetrahedrons have been included. New result is that L points are inverses of each other with respect to the circumscribed circle (sphere in 3D).

**Keywords:** Classical geometry, visualization, Euler line, triangle, tetrahedron

## 1. Introduction

Goal of this paper is to show the significant value of the program GeoGebra in teaching geometry, in visual understanding of problems, and emphasize the basic role of the elementary approach.

GeoGebra is a geometric and algebraic tool basically intended for teaching at schools and universities. After an easy registration, the user can quickly start experimenting. Basic mode is when there is an algebraic and a drawing window. Geometrical elements can be shown in the geometrical window. It is suitable for two- and three dimensions as well. Numerical data of objects are found in the algebraic window. The program is easily usable, the clear and thorough approach can be enjoyed. This program can also be used for visualizing and solving rather complex research problems.

In this paper, our method is the following. Applying GeoGebra, a strong conjecture is found and then it is proved analytically. This idea is quite essential, because physical applications can also appear.

Formerly, mathematics was a deductive science, in contrary to physics that is a fully inductive science based on experiments. Formerly, in mathematics, there were no experiments. With appearance of DGS, experiments became possible in geometry as well. Thus, geometry is currently becoming partly an inductive science [4]. This idea is not new. In other branches of mathematics, experiments started to become possible with appearance of computer.

## 2. Two-dimensional problems

### 2.1. The new notable points (L points) in GeoGebra

First, we reproduce in elementary way, statements of our paper [2]. The elementary solution leads to Apollonius circles. Intersections of three Apollonius circles should be drawn.

The problem is as follows. Given a general triangle $ABC$. It is to find point $X$ satisfying the following equation:

$$\frac{AX}{BC} = \frac{BX}{CA} = \frac{CX}{AB} \tag{2.1.1}$$

where $AX$ denotes the segment between $A$ and $X$, and so on.

In general, $\frac{AX}{BX} = const.$ determines the Apollonius curve belonging to the given $A$, $B$ and the constant. For example, it is well known that locus of points for the $ABC$ triangle satisfying $\frac{AX}{BX} = \frac{AC}{BC}$, is an Apollonius circle. Easy to show that this Apollonius circle intersects the circumscribed circle k perpendicularly. It is also well known that for any two vertices of the triangle, and for the ratio of the opposite edges, there is an Apollonius circle, and these circles intersect each other in two common points.

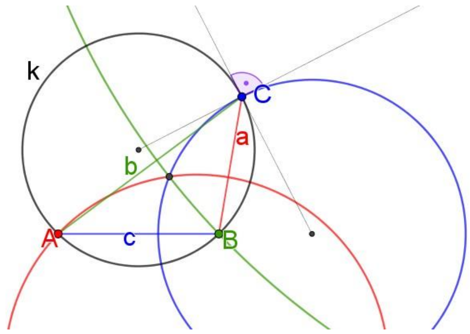


*Fig. 2.1.1. The well-known Apollonius circles. Corresponding elements are denoted with the same color*

In our case, however, locus of points $X$ should be found that belong to $\frac{AX}{BX} = \frac{BC}{AC}$. These are Apollonius circles of triangles $A'BC$, $AB'C$ and $ABC'$, where $A'$, $B'$, $C'$' are mirrors of $A$, $B$, $C$ with respect to perpendicular bisectors belonging to the opposite sides.

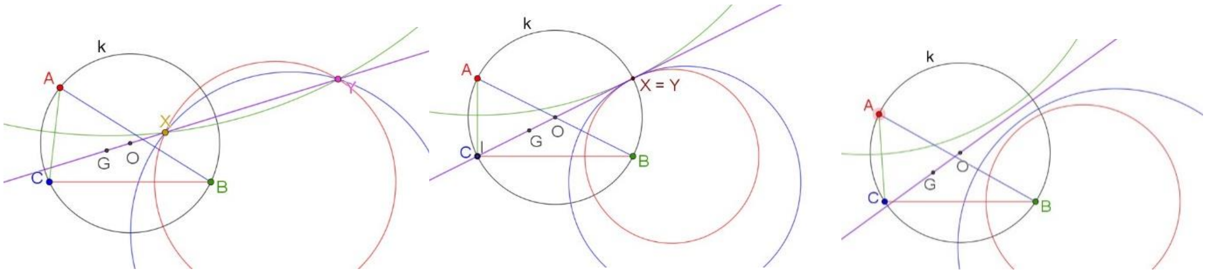


*Fig. 2.1.2. From left to right, number of L points: 2, 1 or 0. Notations: k – circumscribed circle, O – center of the circumscribed circle, G – centroid, X, Y – L points*

Drawing these circles, and exploiting the dynamic tools of GeoGebra, we can recognize when $A$, $B$, $C$ are moved, that

1. These circles also perpendicularly intersect the circumscribed circle of $ABC$,
2. When they intersect each other, then these three circles have 2 common points,
3. When two of them touch each other than the third one touch at the same point,
4. When they have no common points, they have a common power line,
5. Their power line is the Euler line of the triangle

These statements are analyzed in the following.

1. For example, $C'$ also fits to the circumscribed circle k, thus, the circle belonging to $\frac{AX}{BX} = \frac{BC}{AC} = \frac{AC\prime}{BC\prime}$ also perpendicular to $k$.
2. This fitting follows from the transitivity of the mentioned equation, thus if the L point fits to two circles, then it fits to the third one as well,
3. If two circles touch each other, then, as they are perpendicular to k, the touching point fits to k. This occurs if and only if $ABC$ is a right triangle. In this case the L points coincide, and they are mirrors of the right-angle vertex to the center of the opposite edge,
4. This statement is proved analytically in [1]. Proof with use of elementary geometrical tools is an open problem.

## 2.2. Inversion

As we mentioned at the beginning, first we show with GeoGebra that the L points are inverses of each other with respect to the circumscribed circle. This is a strong conjecture shown in Fig. 2.2.1. We calculated it numerically with 15-digit accuracy that inversion holds.

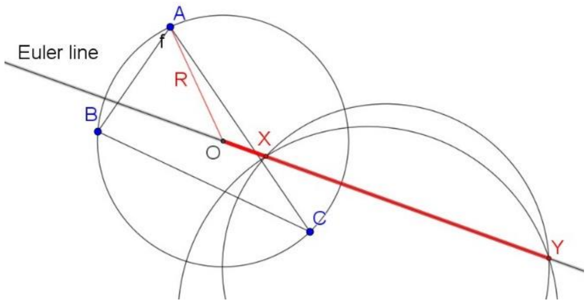


*Fig. 2.2.1. Conjecture that the L points are inverses of each other with respect to the circumscribed circle. Reason of the conjecture is that OX*OY=R² with 15-digit accuracy*

This statement is a consequence of points 1 and 2 above, with a theorem that if two circles intersect each other perpendicularly, then one of them is invariant with respect to the inversion concerning the other one, or in other words, if the circle $s$ is perpendicular to the circle $k$, then inverse of any point of s also fits to $s$. This relation, as perpendicularity between two circles or lines, is symmetric.

"A mathematical relation should be proved many ways not because one is more convincing than the other, but because different proofs enlighten different aspects of the same relation." In this sense, the other proofs follow below.

Given a triangle $ABC$ and the notable point $X$. We define $X$ as follows:

$$\sqrt{\mu_1} = \frac{XA}{BC} = \frac{XB}{CA} = \frac{XC}{AB} \tag{2.2.1}$$

**Theorem 1**: If $Y$ is an inversion of $X$ with respect to the circumscribed circle, then $Y$ is also a notable point.

**Theorem 2**: If $X$ and $Y$ are notable points in the sense of (1), then they are inverses of each other, with respect to the circumscribed circle.

**Proof of Theorem 1**: Denote $W$ the intersection of $XY$ and the circumscribed circle. $W, X, Y$ fit to the same line across the center of the circumscribed circle because $Y$ is an inversion of $X$:

$$\underline{x} = \underline{z} + \mu_1 \underline{v} \tag{2.2.2}$$

$$\underline{y} = \underline{z} + \mu_2 \underline{v} \tag{2.2.3}$$

$$\underline{w} = \underline{z} + \mu_3 \underline{v} \tag{2.2.4}$$

where $Z$ is the center of the circumscribed circle and $\underline{v}$ is a vector of arbitrary length that satisfies the requirement for the Euler line. $Y$ is the inversion of $X$:

$$\underline{xy} = R^2 \tag{2.2.5}$$

where $R$ is the radius of the circumscribed circle:

$$R = |\underline{a} - \underline{z}| = |\underline{b} - \underline{z}| = |\underline{c} - \underline{z}| = |\underline{w} - \underline{z}| \tag{2.2.6}$$

From (2.2.5):

$$\mu_1 \mu_2 = \mu_3^2 \tag{2.2.7}$$

We should prove that

$$\sqrt{\mu_2} = \frac{YA}{BC} = \frac{YB}{CA} = \frac{YC}{AB} \tag{2.2.8}$$

Method of proof is that we assume (2.2.8) and we show that it leads to a right statement.

It is to prove:

$$\frac{|\underline{x}-\underline{a}|}{\sqrt{\mu_1}} = \frac{|\underline{y}-\underline{a}|}{\sqrt{\mu_2}} \tag{2.2.9}$$

where from (2.2.2-3):

$$\frac{\mu_1}{\mu_2} = \frac{|\underline{x}-\underline{z}|}{|\underline{y}-\underline{z}|} \tag{2.2.10}$$

$$\left(\underline{x} - \underline{a}\right)^2 - \frac{\mu_1}{\mu_2}\left(\underline{y} - \underline{a}\right)^2 = 0 \tag{2.2.11}$$

$$\left(\underline{x} - \underline{a} + \sqrt{\frac{\mu_1}{\mu_2}}\left(\underline{y} - \underline{a}\right)\right)\left(\underline{x} - \underline{a} - \sqrt{\frac{\mu_1}{\mu_2}}\left(\underline{y} - \underline{a}\right)\right) = 0 \tag{2.2.12}$$

It is suspected that $\underline{x} - \underline{a} + \sqrt{\frac{\mu_1}{\mu_2}}\left(\underline{y} - \underline{a}\right) \perp \underline{x} - \underline{a} - \sqrt{\frac{\mu_1}{\mu_2}}\left(\underline{y} - \underline{a}\right)$.

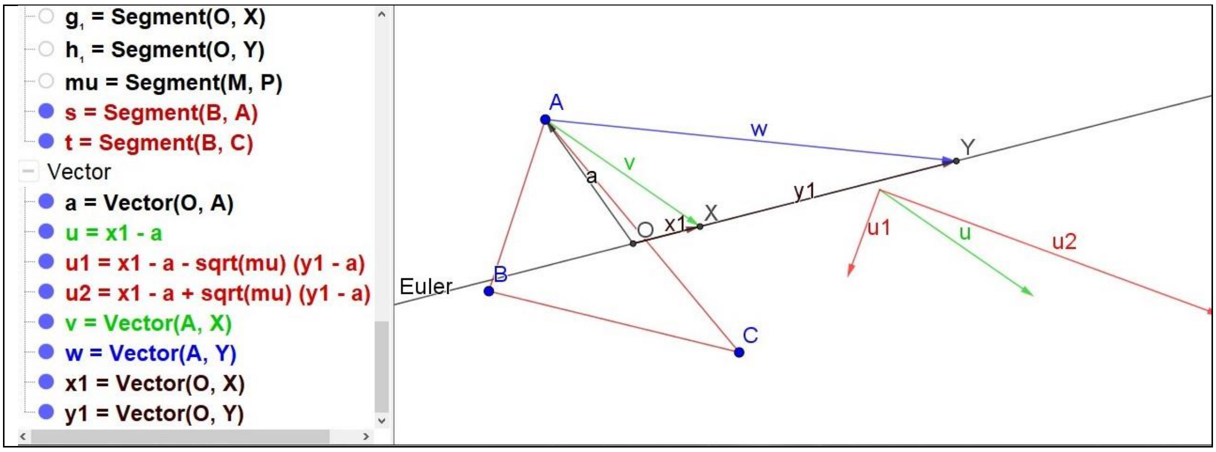


*Fig. 2.2.2. Perpendicularity condition in GeoGebra. We are speaking about u1 and u2*

We can easily see that the triangle with edges $\left|\underline{x} - \underline{a} + \sqrt{\frac{\mu_1}{\mu_2}}\left(\underline{y} - \underline{a}\right)\right|$, $\left|\underline{x} - \underline{a} - \sqrt{\frac{\mu_1}{\mu_2}}\left(\underline{y} - \underline{a}\right)\right|$, $\left|2\sqrt{\frac{\mu_1}{\mu_2}}\left(\underline{y} - \underline{a}\right)\right|$ is a right triangle. Because imagine the triangle with vertices (0,0), $2\sqrt{\frac{\mu_1}{\mu_2}}\left(\underline{y} - \underline{a}\right)$ and $\underline{x} - \underline{a} + \sqrt{\frac{\mu_1}{\mu_2}}\left(\underline{y} - \underline{a}\right)$, and easy to see that these vertices are on a Thales circle with origin $\sqrt{\frac{\mu_1}{\mu_2}}\left(\underline{y} - \underline{a}\right)$ and radius $|\underline{x} - \underline{a}|$. The proof is similar when $\underline{a}$ is replaced by $\underline{b}$ or $\underline{c}$. Thus Theorem 1 is proved.

**Proof of Theorem 2**:

Conditions: $X$ and $Y$ satisfy (2.2.1) and (2.2.8).

Then they fit to the Euler line, that are (2.2.2) and (2.2.3), with proper choice of the direction of $\underline{v}$.

Then the intersection of $XY$ and the circumscribed circle, $W$, also fit to the Euler line of (2.2.4).

We should prove that (2.2.5) holds:

$$\underline{x}\underline{y} = R^2 \tag{2.2.13}$$

Now we substitute (2.2.2-3) into (2.2.13):

$$\mu_1\mu_2(\underline{v})^2 = R^2 \tag{2.2.14}$$

Now we choose $\underline{v}$ so that

$$|\underline{v}| = R \tag{2.2.15}$$

Then (14) becomes

$$\mu_1\mu_2 = 1 \tag{2.2.16}$$

Then we consider that $\mu_1$ and $\mu_2$ are the two solutions of an equation for $\mu$.

We start from (2.2.2). Then we put the origin to the center of the circumscribed circle, then (2.2.2) becomes:

$$\underline{x} = \mu\underline{v} \tag{2.2.17}$$

Substituting (17) into (1):

$$(\mu\underline{v} - \underline{a})^2 = \mu(\underline{b} - \underline{c})^2 \tag{2.2.18}$$

$$\mu^2R^2 - 2\mu\underline{v}\underline{a} + |\underline{a}|^2 - \mu|\underline{b} - \underline{c}|^2 = 0 \tag{2.2.19}$$

$$\mu^2 - \mu\frac{2\underline{v}\underline{a}}{R^2} + 1 - 2\mu + \mu\frac{2\underline{b}\underline{c}}{R^2} = 0 \tag{2.2.20}$$

$$\mu^2 - \mu\frac{2\underline{v}(\underline{a}+\underline{b}+\underline{c})}{3R^2} + \mu\frac{2(\underline{b}\underline{c}+\underline{a}\underline{c}+\underline{a}\underline{b})}{3R^2} - 2\mu + 1 = 0 \tag{2.2.21}$$

$$(\mu - \mu_1)(\mu - \mu_2) = \mu^2 - \mu(\mu_1 + \mu_2) + \mu_1\mu_2 = 0 \tag{2.2.22}$$

Comparison of (2.2.21) and (2.2.22) yields

$$\mu_1\mu_2 = 1 \tag{2.2.23}$$

That is, we wanted to prove.

## 3. Three-dimensional problems

Here we mention those problems that are left open in [3].

### 3.1. Case of two L points in tetrahedrons

In 3d, the L points can be found by intersection of Apollonius spheres. In Fig. 3.1. we show the case with two solutions. Solutions are depicted here as intersections of main circles of the Apollonius spheres.

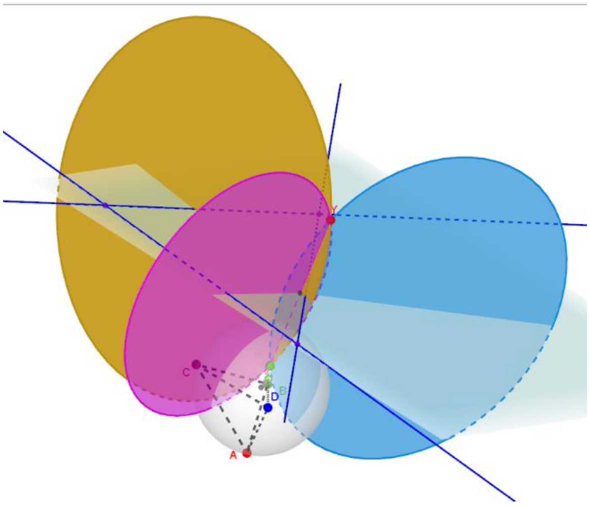


*Fig. 3.1. Case of two solutions in 3d*

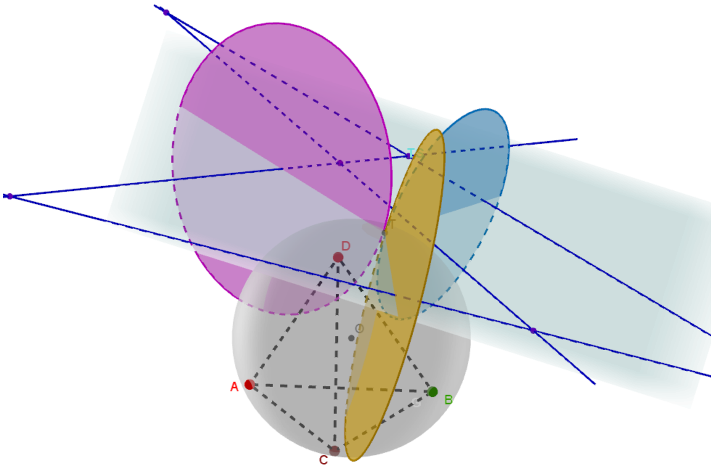


*Fig. 3.2. One solution*

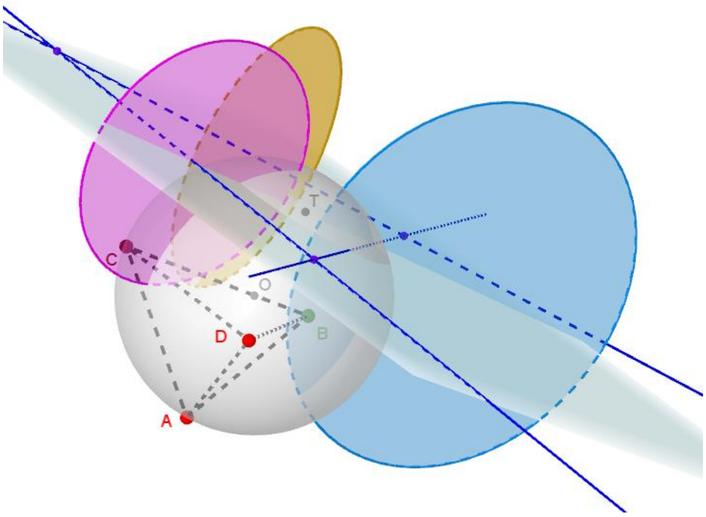


*Fig. 3.3. Zero solution*

Based on some GeoGebra runs, the following conjecture can occur. If two L points exist, then one is the inversion of the other with respect to the circumscribed sphere. Reason of this conjecture is also an experiment in GeoGebra.

**Definition 3.1:** Given the tetrahedron ABCD where $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ are the position vectors pointing to the vertices. The L point $X$ is defined as

$$\frac{|\underline{x}-\underline{a}|^2}{A_{bcd}^2} = \frac{|\underline{x}-\underline{b}|^2}{A_{cda}^2} = \frac{|\underline{x}-\underline{c}|^2}{A_{dab}^2} = \frac{|\underline{x}-\underline{d}|^2}{A_{abc}^2} \tag{3.1}$$

where $A_{abc}$ denotes the area of the face $ABC$, and $\underline{x}$ is a position vector to the point X.

**Theorem 3.1:** If $X$ exists and $Y$ is the inversion of $X$ with respect to the circumscribed sphere, then $Y$ is an L point.

**Proof:** Along the lines of the two-dimensional case in Section 2.2.

### 3.2. Case of one solution

This occurs when the plane of the centers of the Apollonius spheres touches the circumscribed sphere. Graphically this case is shown in Fig. 3.2. Alternative sufficient conditions are found in [3].

### 3.3. Case of zero solution, condition of the existence of the solution

This case is shown in Fig. 3.3.

The critical point is when the plane of the centers of the Apollonius spheres touches the circumscribed sphere. In case of no intersection, there is no solution, in case of intersection, there are two solutions. One solution occurs when there is a touch. In the following, the three cases are visualized. Then it is to find what the condition is for the vertices of the tetrahedron.

## 4. Conclusions

Our intention was to show versatility of the program GeoGebra. It is a strong help in research as well, as we have shown it in examples. The authors hope sincerely that this contribution obtains a further help in making GeoGebra even more widespread.

A possible continuation may be a further investigation on the condition of existence of the L point in terms of vertex coordinates. Essential applications are expected in microwave measurement techniques and properties of tetrahedron-like molecules. Also, there is a possibility in telecommunications: Tetrahedron-like nets of mobile phones.

## 5. References

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Lajos Szilassi is a retired geometer at the University of Szeged. His best result is the Szilassi polyhedron, a geometrical body whose all faces are neighbors of all other faces. He has a DSc from the Hungarian Academy of Sciences.

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