



Vector Space Definition in *Vector Spaces* Chapter

Vector Space Definition Link: <https://www.geogebra.org/m/XnfUWvvp#material/EkhJqaFE>

Learning Goals:

Students will:

- Use the GeoGebra applet **Vector Space Definition** to support the understanding of the properties required to define a vector space. Express and apply vector properties algebraically and graphically by hand and with the interactive GeoGebra applet **Vector Space Definition**.

The video provides a visual introduction of the GeoGebra applet **Vector Space Definition**.

Activities: All activities reference the GeoGebra applet **Vector Space Definition**.

Note: the properties are illustrated using 2D vectors, but these properties equally apply to any dimensional vectors. For example, look at the applet **Associativity Property-3D** which illustrates the associative property with three-dimensional vectors. It is in the *Vector Spaces* chapter.

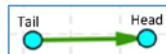
Property 1: the set V of vectors is closed under vector addition, that is, $\mathbf{u} + \mathbf{v} \in V$

Example: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

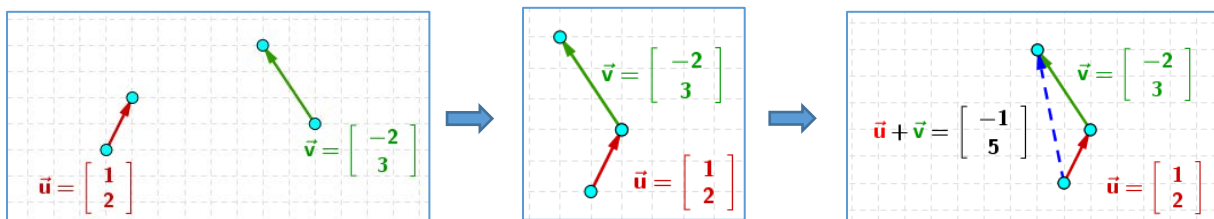
Vector addition is performed algebraically by adding each corresponding component of the vectors, which creates a new vector.

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

This is a graphical view of a vector.



Graphically, vector addition is represented by connecting the tail of one vector to the head of another vector. This results in the *new vector* whose *tail* is the tail of the first vector \mathbf{u} and whose *head* is the head of the second vector \mathbf{v} which was added to the first.





To illustrate this property with the GeoGebra applet grab near the middle of the vector to move it so that its tail is connected to the head of the other vector.

Why does $\mathbf{u} + \mathbf{v}$ create a vector?

In following three examples, algebraically add the vectors \mathbf{u} and \mathbf{v} to get the vector $\mathbf{u} + \mathbf{v}$ in algebraic form. Then using the GeoGebra applet, geometrically add the vectors, creating the vector $\mathbf{u} + \mathbf{v}$ in graphical form.

In the applet, a vector can be changed by grabbing and moving its head or tail. When vectors are properly added, a blue dotted vector, the addition vector, will appear with the answer.

1. If $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, what is $\mathbf{u} + \mathbf{v}$?

Calculate it algebraically:

Geometrically add the vectors using the applet, then paste your result here:

2. If $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$, what is $\mathbf{u} + \mathbf{v}$?

Calculate it algebraically:

Geometrically add the vectors using the applet, then paste your result here:

3. If $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$, what is $\mathbf{u} + \mathbf{v}$?

Calculate it algebraically:

Geometrically add the vectors using the applet, then paste your result here:



Property 2: vector addition is associative, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

With this property in the GeoGebra applet, the vectors at the top will define the vectors on the bottom. Unless you want to change the vector definition, do not alter the vectors on the top.

For this property a vector is drawn as follows:

The tail is used to move these vectors.



When demonstrating this property the vectors on the bottom are moved. Do the additions listed. First do

Add : $(\vec{u} + \vec{v}) + \vec{w}$ Note, the addition in parentheses is done first, then the other vector is added.

When each addition is completed, a new dotted vector will appear having the label of that addition. (This dotted line should look familiar, as it represents the addition vector just like the ones that were found in Property 1.)

Paste the graphical results of $(\vec{u} + \vec{v}) + \vec{w}$ here:

Now do

Add : $\vec{u} + (\vec{v} + \vec{w})$ Note, again, the addition in parentheses is performed first, then the other vector is added to that resulting vector.

Paste the graphical results of $\vec{u} + (\vec{v} + \vec{w})$ here:

If $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, show using algebra that $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$. Use the template below to walk through this process step by step by filling in the blanks.

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$\left(\begin{bmatrix} \\ \end{bmatrix} + \begin{bmatrix} \\ \end{bmatrix} \right) + \begin{bmatrix} \\ \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix} + \left(\begin{bmatrix} \\ \end{bmatrix} + \begin{bmatrix} \\ \end{bmatrix} \right)$$



$$\left(\begin{bmatrix} \\ \end{bmatrix} \right) + \begin{bmatrix} \\ \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix} + \left(\begin{bmatrix} \\ \end{bmatrix} \right)$$
$$\begin{bmatrix} \\ \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

Now, using the GeoGebra applet, create these vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ as the given vectors. Perform the two add operations. When done, make a screen copy of the left window and paste it here:

Property 3: there is a zero vector $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$

A vector can have components consisting of any real numbers, including zero.

In this property, we see that the zero vector is defined as $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We will examine what that means.

Given $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, show what $\mathbf{v} + \mathbf{u}$ looks like algebraically; check your answer using the GeoGebra applet to add the two vectors. What vector is your result?

Paste a copy of the window where \mathbf{v} and $\mathbf{0}$ have not been added.



Now, paste a copy of the window where \mathbf{v} and $\mathbf{0}$ have been added.

Why is the $\mathbf{0}$ vector called the additive identity?

Property 4: each $\mathbf{v} \in V$ has an additive inverse $\mathbf{w} \in V$ such that $\mathbf{w} + \mathbf{v} = \mathbf{0}$

Since vectors are defined with numeric components and every number has an additive inverse (e.g. $3 + (-3) = 0$), adding vectors is just adding the corresponding numeric components. Therefore it is reasonable that two vectors with corresponding components which are additive inverses will geometrically be added such that one vector plus the other gets you back to the start point, creating a zero vector appearing as a single point.

In the GeoGebra applet, this property has vectors just like in Property 1. You will need to alter vectors by grabbing the head or tail. To see if you have created the additive inverse, you will need to grab the vector near the middle and drag it so that head of one connected to the tail of the other.

When the vector \mathbf{v} is added to its inverse \mathbf{w} , what vector do you get? Write it here:

Show this property algebraically with $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$. After you have something written down, use the GeoGebra applet to see if you got the correct answer.

Paste from the worksheet that process here.



Give a couple of other \mathbf{u} and \mathbf{v} definitions below and algebraically add them. Work with the GeoGebra applet to show that you are correct; paste your results here. At least one example should have \mathbf{u} defined with one positive integer and one negative integer.

$$\mathbf{u} = \begin{bmatrix} \quad \\ \quad \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} \quad \\ \quad \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} \quad \\ \quad \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Why does this make sense?

Property 5: vector addition is commutative, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

The vectors in this vector space property are similar to those in Property 1 and Property 2.

If $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, algebraically what is $\mathbf{v} + \mathbf{u}$? What is $\mathbf{u} + \mathbf{v}$? Show your work.

Use the GeoGebra applet to find the vector value of each side of the equation and paste your results here.



Property 6: the set V is closed under scalar multiplication, that is, $r \cdot \mathbf{v} \in V$

Since we now know that vectors have numeric properties when added or subtracted, we should start to think about what vectors do with multiplication. Just like adding, multiplication can be applied over a vector with scalars. This scalar will be multiplied to all of the components in the vector:

$$\text{If } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } r = 3$$

$$3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 \\ 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Try this in the GeoGebra applet by using the $r +$ symbol.

You can change the \mathbf{v} vector by grabbing one of the large blue dots on the vector on the right. This will automatically change the scalar vector on the left. Show the algebraic process of these scalar products. Double check your work on the GeoGebra applet.

$$\mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}; r = 2$$

$$\mathbf{v} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}; r = -1$$

$$\mathbf{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}; r = 3$$

Property 7: addition of scalars distributes over scalar multiplication,

$$(r + s) \cdot \mathbf{v} = r \cdot \mathbf{v} + s \cdot \mathbf{v}$$

In this property, make sure to remember what you have learned about vector addition. The defined \mathbf{v} vector is off to the left, and the scalars added then multiplied by \mathbf{v} is in the middle, and \mathbf{v} multiplied by each scalar and added together is on the right. Since vector addition is lining up vectors head to tail, the vectors on the right might look a little strange at times.

For example, in the GeoGebra applet, set $r = -1$ and $s = 1$. This should yield the zero vector according to the equation on the top. You'll see that the vectors on the right has two arrow heads. Why is this?



You should have noted that you still get the zero vector, it's just getting to it by adding two equal and opposite vectors together! Now take the time to change the scalars and even alter the definition of \mathbf{v} by grabbing one of the large blue dots on either side of it.

Algebraically show this property if $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $r = 2$, $s = 1$. Prove your answer using the applet.

Property 8: scalar multiplication distributes over vector addition,

$$r \cdot (\mathbf{v} + \mathbf{w}) = r \cdot \mathbf{v} + r \cdot \mathbf{w}$$

Just like in the previous property, we can examine multiplication over vectors. This property combines Property 1 and Property 6, in that we can not only add vectors but we can multiply scalars at the same time!

With this property, you can redefine vectors \mathbf{v} and \mathbf{w} by dragging the large blue dots at the end of either vector. You will notice that all of these vectors will 'stick' together to make this GeoGebra property easier to play with. The pink vector represents the addition vector of $\mathbf{v} + \mathbf{w}$.

Define $\mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. What is the addition vector if $r = 1$?

Use this step by step template to show why this is true if $r = 2$. Do the algebra first and check your answer against the results in the applet.

$$(\quad) \cdot \left(\begin{bmatrix} \quad \\ \quad \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \end{bmatrix} \right) = (\quad) \cdot \begin{bmatrix} \quad \\ \quad \end{bmatrix} + (\quad) \cdot \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$$\left(\begin{bmatrix} \quad \\ \quad \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \end{bmatrix} \right) = \begin{bmatrix} \quad \\ \quad \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$$\begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$



Property 9: ordinary multiplication of scalars associates with scalar multiplication,
$$(rs) \cdot \mathbf{v} = r \cdot (s \cdot \mathbf{v})$$

This property shows us that the order of the multiplication can change, but you should arrive at the same answer. If there is more than one scalar multiple being put onto a vector, we can choose to multiply the scalars together first and then multiply that single scalar by the vector or we can multiply them by the vector one after the other. When you open the GeoGebra applet, the \mathbf{v} vector is set as $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and r and s are set as 1.

Redefine $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ by grabbing the large blue dots on either end of the vector on the far left. You'll see that this will automatically update the other two vectors. Observe and make notes on what happens as you add and subtract from r and s . Make sure you observe negatives and 0.

Property 10: multiplication by scalar 1 is the identity operation, $1 \cdot \mathbf{v} = \mathbf{v}$