KINEMATICS AND PROJECTILE MOTION

AN ATTEMPT AT A UNIFIED APPROACH

INTRODUCTION

It may seem that the several equations used to describe the various kinds of motion we have studied are unrelated. If we re-examine these motions with a consistent, mathematically careful notation, we will see that this apparent confusion vanishes. Indeed, much of the confusion arises in the attempt to avoid mathematical notation that is, admittedly, at first glance, cumbersome. However, the fact cannot be avoided that **mathematics is the language of physics**.

We will try to unify our descriptions of (1) one-dimensional motion (often called "horizontal" motion, though it need not actually be horizontal); (2) free fall; (3) vertical motion; (4) projectile (two-dimensional) motion at an acute angle; (5) horizontal projectile motion.

The analysis begins with the assumption that the **acceleration is constant.** This most fundamental fact leads to a mathematical description called a "second-order differential equation." When this equation is solved, using calculus, we get two equations of motion-- one for the velocity, and another for the position. Both of these equations of motion are continuous functions of time. These two equations are the starting point for all of our analysis of motion.

Writing these equations of motion for the two-dimensional case, in which we have also assumed that the **motions in the *x* and *y* directions are independent**, we have:

$$x(t) = x(0) + v_x(0)t + \frac{1}{2}a_x t^2 \qquad v_x(t) = v_x(0) + a_x t \tag{1}$$

$$y(t) = y(0) + v_y(0)t + \frac{1}{2}a_y t^2 \qquad v_y(t) = v_y(0) + a_y t \tag{2}$$

All the other kinematic equations that we have are derived from these; for example, by solving the velocity equation for time, and using that in the position equation. We derived these various equations using geometric arguments (areas of trapezoids, etc.), but this can all be done with algebra.

In the one-dimensional case, we used the equivalent of Eq(1), but with the "delta" notation. That notation obscures the continuous nature of the variation in *x* or *v* with time. Also, in one-dimensional vertical motion, the delta notation can cause confusion about the starting and ending y position. Thus we will now write out the functions of time explicitly.

Next we consider the various kinds of motion, using the same notation for each. Then, the connections between them should become clear.

ONE-DIMENSIONAL MOTION (OTHER THAN VERTICAL)

The position vs. time and velocity vs. time equations are given above as Eq(1). We use "*x*" as our position variable, but it need not be the horizontal direction, which we usually denote as "*x*". The acceleration is of course constant, and may or may not be zero. We solved these problems using the "recipe" approach, where we listed the variables and selected from the four kinematic equations. Two of those four are in Eq(1), and the other two can be derived from them. One of the latter that is often useful is

$$v_t^2 = v_0^2 + 2a\left(x_t - x_0\right) \tag{3}$$

Here we introduce the use of a subscript "t" in place of the parentheses, so that v_t is the same as *v(t)* and v_0 is the same as *v(0).* This just makes the equations a bit easier to read. This notation makes it clear that the equation is a function of time, although time does not appear in it explicitly.

TWO-DIMENSIONAL MOTION: PROJECTILE MOTION

In this case, we have a few new facts to consider, and these facts cause changes in the kinematic equations. First, we need to resolve the initial velocity vector into its vertical and horizontal components. Second, there is no acceleration in the horizontal direction. Third, the acceleration in the vertical direction is (almost) always - g. Fourth, we can always define our coordinate system in such a way that the initial *x* position is zero. Using these facts in Eq(1) we can then write a new set of equations of motion:

$$x(t) = x_0 + v_0 \cos(\theta)\, t + \frac{1}{2} a_x t^2 \qquad v_x(t) = v_0 \cos(\theta) + a_x t$$

which simplify to

$$x(t) = v_0 \cos(\theta)\, t \qquad v_x(t) = v_0 \cos(\theta) \tag{4}$$

$$y(t) = y_0 + v_0 \sin(\theta) t - \frac{1}{2} g t^2 \qquad v_y(t) = v_0 \sin(\theta) - g t \tag{5}$$

Equation sets (4) and (5) are the fundamental equations of motion for projectiles.

It happens that Eq(3) adapted to the vertical direction is particularly useful, so here it is:

$$v_y(t)^2 = \left[v_0 \sin(\theta)\right]^2 - 2 g \left(y_t - y_0\right) \tag{6}$$

Again, note that time does not appear explicitly in Eq(6), but is there implicitly, since the velocity on the left hand side is evaluated at whatever time *y* had the value y_t. This equation is well-suited for problems involving finding the maximum height, since we know that the vertical velocity is zero there.

We have stated that this form of motion is restricted to acute angles, and we note also that the initial *y* position is often taken to be zero. However, if y_0 is not zero we could have a negative angle, that is, the object could be thrown downward from some elevation These equations apply to that situation, without modification (the sine of an angle in the fourth quadrant will be negative, so the velocity is negative).

VERTICAL MOTION

In this special case of projectile motion we have a zero velocity in the *x* direction, and the angle at which the object is propelled is of course 90 degrees. This leads to the modified equations of motion:

$$x(t) = 0 \qquad v_x(t) = 0 \tag{7}$$

$$y(t) = y_0 + v_0 t - \frac{1}{2} g\, t^2 \qquad v_y(t) = v_0 - g t \tag{8}$$

Note that the sine drops out of Eq(5) to give Eq(8), since the sine of 90 degrees is one. Eq(7) follows from Eq(4) since the cosine of 90 degrees is zero. Also, we often choose y_0 to be zero, but it need not be.

FREE-FALL MOTION

This is a special case of vertical motion, in which the object is simply released with zero initial velocity, and it accelerates downward. In this case, clearly the initial *y* cannot be zero. The equations of motion are again Eq(7) for *x*, and for *y* we have

$$y(t) = y_0 - \frac{1}{2} g t^2 \qquad v_y(t) = -g t \tag{9}$$

HORIZONTAL PROJECTILE MOTION

This is a special case of projectile motion, in which the object is propelled horizontally, that is, at an angle of zero degrees. Then we can adapt Eq(4) and (5) to give the equations of motion for this case:

$$x(t) = v_0 t \qquad v_x(t) = v_0 \tag{10}$$

$$y(t) = y_0 - \frac{1}{2} g t^2 \qquad v_y(t) = -g t \tag{11}$$

Note how the various terms in Eq(4) and (5) are adjusted due to the sine and cosine of zero degrees. Also notice that in this form of motion the initial *y* must not be zero. Comparing to the free-fall case, it is apparent from the equations that the motion in the *y*-direction is the same. This is due to the assumed independence of the *x* and *y* motions. (By contrast, when air effects are considered, the motions are no longer independent, and the description of motion requires more advanced mathematics.)

FACTS + EQUATIONS = SOLUTIONS

Here are some facts and/or assumptions that we can use with these equations to find solutions.

1. For projectile motion at an acute angle the vertical velocity at the maximum height is zero. This fact can be used to find the maximum height, via Eq(6).

2. If the value of *y(T),* where *T* is the "time of flight" (TOF), is equal to the initial *y*, then the trajectory is symmetric, and it can easily be shown that the time to reach the maximum height is half the TOF.

3. For many problems, *y(T)* = 0. This can be used with Eq(5) in finding solutions.

4. The "range" (R) is defined to be the maximum horizontal position attained, such that *x(T) = R*. Since the motion in the *x* direction is not accelerated (by assumption- this is not in fact always true), we can find the range using

   $$R = v_0 \cos(\theta) T \tag{12}$$

   Note that the range has no meaning for vertical and free-fall motion; the cosine of 90 degrees is zero, so the range is zero.

DERIVED EQUATIONS

In an earlier document the following equations were derived; we repeat them here for reference:

$$T = \frac{2 v_0}{g} \sin(\theta) \qquad R = \frac{v_0^2}{g} \sin(2\theta) \qquad y_{\max} = y(t_{\max}) = y\left(\frac{T}{2}\right) = \frac{v_0^2}{2g} \sin^2(\theta)$$

Beware: these equations only apply to projectile motion with a symmetric trajectory-- *y(T)* = y_0!

EQUATION SUMMARY

GENERAL 2D KINEMATICS **includes 1D motion**

$$x(t) = x(0) + v_x(0)t + \frac{1}{2}a_x t^2 \qquad v_x(t) = v_x(0) + a_x t$$

$$y(t) = y(0) + v_y(0)t + \frac{1}{2}a_y t^2 \qquad v_y(t) = v_y(0) + a_y t$$

PROJECTILE **acute or negative angle**

$$x(t) = v_0 \cos(\theta)\, t \qquad v_x(t) = v_0 \cos(\theta)$$

$$y(t) = y_0 + v_0 \sin(\theta) t - \frac{1}{2} g t^2 \qquad v_y(t) = v_0 \sin(\theta) - g t$$

VERTICAL **angle = 90 degrees**

$$x(t) = 0 \qquad v_x(t) = 0$$

$$y(t) = y_0 + v_0 t - \frac{1}{2} g\, t^2 \qquad v_y(t) = v_0 - g t$$

FREE-FALL **vertical with zero initial velocity**

$$x(t) = 0 \qquad v_x(t) = 0$$

$$y(t) = y_0 - \frac{1}{2} g t^2 \qquad v_y(t) = -g t$$

HORIZONTAL PROJECTILE **angle = 0 degrees**

$$x(t) = v_0 t \qquad v_x(t) = v_0$$

$$y(t) = y_0 - \frac{1}{2} g t^2 \qquad v_y(t) = -g t$$

USEFUL FACTS

$$T = 2\, t_{\max} \;\text{ iff }\; y(T) = y_0 \qquad R = v_x T = v_0 \cos(\theta) T \qquad v_y(t_{\max}) = 0$$

$$v_y(t)^2 = \left[v_0 \sin(\theta)\right]^2 - 2 g \left(y_t - y_0\right)$$