INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

MAA

EXERCISES [MAA 3.5]

SIN, COS, TAN ON THE UNIT CIRCLE – IDENTITIES

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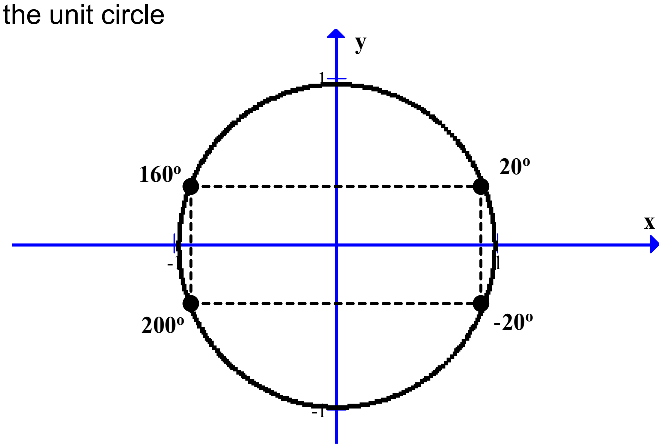
O. Practice questions

1. [Maximum mark: 12] ***[without GDC]***

Let $\sin 20° = p$, and $\cos 20° = q$ (so that $p^2 + q^2 = 1$)

By observing the unit circle



express the following in terms of p **and/or** q

$\sin 160°$		$\sin 200°$	
$\cos 160°$		$\cos 200°$	
$\tan 160°$		$\tan 200°$	

$\sin 340°$		$\sin(-20°)$	
$\cos 340°$		$\cos(-20°)$	
$\tan 340°$		$\tan(-20°)$	

2. [Maximum mark: 5] ***[without GDC]***

Let $\sin 20° = p$, and $\cos 20° = q$ (so that $p^2 + q^2 = 1$)

Write down expressions for the following

$\tan 20^{o} =$		(in terms of p and q)
$\sin 40^{o} =$		(in terms of p and q)
$\cos 40^{o} =$		(in terms of p and q)
		(in terms of p **only**)
		(in terms of q **only**)

3. [Maximum mark: 7] ***[without GDC]***

Given that $\sin x = p$, where x is an **acute** angle

(a) Find the value of $\cos x$ in terms of p; [2]

(b) **Hence**, express the following in terms of p:

(i) $\tan x$ (ii) $\cos 2x$ (iii) $\sin 2x$ (iv) $\tan 2x$ (v) $\sin 4x$ [5]

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	Formula	**Expression in terms of p**
$\tan x$		
$\cos 2x$		
$\sin 2x$		
$\tan 2x$		
$\sin 4x$		

4. [Maximum mark: 15] ***[without GDC]***

Let θ be an angle in the first quadrant, i.e. $0° < \theta < 90°$.

(a) Given that $\sin\theta = \frac{3}{4}$, find (i) $\cos\theta$ (ii) $\tan\theta$. [5]

(b) Given that $\cos\theta = \frac{3}{4}$, find (i) $\sin\theta$ (ii) $\tan\theta$ [5]

(c) Given that $\tan\theta = \frac{3}{4}$, find (i) $\sin\theta$ (ii) $\cos\theta$ [5]

5. [Maximum mark: 15] ***[without GDC]***

Let θ be an angle in the **second** quadrant, i.e. $90° < \theta < 180°$

(a) Given that $\sin\theta = \frac{3}{4}$, find (i) $\cos\theta$ (ii) $\tan\theta$. [5]

(b) Given that $\cos\theta = -\frac{1}{4}$, find (i) $\sin\theta$ (ii) $\tan\theta$. [5]

(c) Given that $\tan\theta = -4$, find (i) $\sin\theta$ (ii) $\cos\theta$. [5]

6. [Maximum mark: 6] ***[without GDC]***

Prove the identities

(a) $\frac{1-\cos 2\theta}{\sin 2\theta} \equiv \tan\theta$. (b) $\frac{\sin 2\theta}{1+\cos 2\theta} \equiv \tan\theta$ [3+3]

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7*. [Maximum mark: 15] ***[without GDC]***

Let θ be an angle for which $\cos\theta + \sin\theta = a$ and $\cos\theta > \sin\theta$.

(a) By taking squares on both sides, find the value of $\sin 2\theta$ in terms of a. [3]

(b) Expand $(\cos\theta - \sin\theta)^2$ and hence find the value of $\cos\theta - \sin\theta$ in terms of a. [3]

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8.** [Maximum mark: 6] ***[without GDC]***

Let θ be an angle for which $\cos\theta + \sin\theta = \frac{4}{3}$.

(a) Find $\sin 2\theta$. [3]

(b) Find $\cos 4\theta$. [3]

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9.** [Maximum mark: 6] ***[without GDC]***

Let θ be an angle for which $\cos\theta - \sin\theta = \frac{1}{2}$.

(a) Find $\sin 2\theta$. [3]

(b) Find $\cos 4\theta$. [3]

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A. Exam style questions (SHORT)

10. [Maximum mark: 4] ***[without GDC]***

Given that $\sin\theta = \frac{1}{2}$, $\cos\theta = -\frac{\sqrt{3}}{2}$ and $0° < \theta < 360°$,

(a) find the value of θ; [2]

(b) write down the **exact** value of $\tan\theta$. [2]

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11. [Maximum mark: 5] ***[without GDC]***

Given that $\sin x = \frac{1}{3}$, where x is an acute angle, find the **exact** value of

(a) $\cos x$; [3]

(b) $\cos 2x$. [2]

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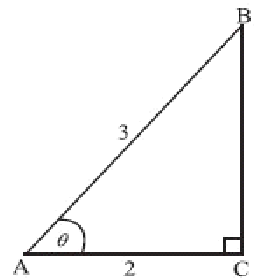
12. [Maximum mark: 6] ***[without GDC]***

The following diagram shows a triangle ABC, where $A\hat{C}B$ is 90°, $AB = 3$, $AC = 2$ and $B\hat{A}C$ is θ.



(a) Show that $\sin 2\theta = \frac{4\sqrt{5}}{9}$. [3]

(b) Find the **exact** value of $\cos 2\theta$. [3]

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13. [Maximum mark: 4] ***[without GDC]***

If A is an obtuse angle in a triangle and $\sin A = \frac{5}{13}$, calculate the value of $\sin 2A$.

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14. [Maximum mark: 6] ***[without GDC]***

Let $p = \sin 40°$, $q = \cos 110°$. Give your answers to the following in terms of p and/or q

(a) Write down an expression for (i) $\sin 140°$; (ii) $\cos 70°$. [2]

(b) Find an expression for $\cos 140°$. [3]

(c) Find an expression for $\tan 140°$. [1]

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15. [Maximum mark: 6] ***[without GDC]***

(a) Given that $\cos A = \frac{1}{3}$ and $0 \le A \le \frac{\pi}{2}$, find $\cos 2A$. [3]

(b) Given that $\sin B = \frac{2}{3}$ and $\frac{\pi}{2} \le B \le \pi$, find $\cos B$. [3]

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16*. [Maximum mark: 7] ***[without GDC]***

The straight line with equation $y = \frac{3}{4}x$ makes an acute angle θ with the x-axis.

(a) Write down the value of $\tan\theta$ [1]

(b) Find the value of (i) $\sin 2\theta$; (ii) $\cos 2\theta$. [6]

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17*. [Maximum mark: 7] ***[without GDC]***

Let $f(x) = \sin^3 x + \cos^3 x \tan x$, $\frac{\pi}{2} < x < \pi$.

(a) Show that $f(x) = \sin x$. [2]

(b) Let $\sin x = \frac{2}{3}$. Show that $f(2x) = -\frac{4\sqrt{5}}{9}$. [5]

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18*. [Maximum mark: 6] ***[without GDC]***

By using the double angle identities for $\cos 2\theta$,

(a) show that $\sin 15^o = \frac{\sqrt{2-\sqrt{3}}}{2}$; [3]

(b) find a similar expression for $\cos 15^o$. [3]

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19*. [Maximum mark: 5] ***[without GDC]***

Prove that $\frac{\sin 4\theta(1-\cos 2\theta)}{\cos 2\theta(1-\cos 4\theta)} = \tan\theta$, for $0 < \theta < \frac{\pi}{2}$, and $\theta \neq \frac{\pi}{4}$.

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20. [Maximum mark: 6] ***[without GDC]***

Given that $2\sin 4x - 3\sin 2x = 0$, and $\sin 2x \neq 0$, find the value of $\cos^2 x$.

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21*. [Maximum mark: 6] ***[without GDC]***

Given that $a\sin 4x + b\sin 2x = 0$, for $0 < x < \frac{\pi}{2}$, find an expression for $\cos^2 x$ in terms of a and b.

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22*. [Maximum mark: 5] ***[without GDC]***

In triangle ABC, AB = 9 cm, AC = 12 cm, and $\hat{B}$ is twice the size of $\hat{C}$.

Find the cosine of $\hat{C}$.

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23*. [Maximum mark: 6] ***[without GDC]***

The triangle ABC has an obtuse angle at B, BC = 10, $\hat{A} = x$ and $\hat{B} = 2x$.

(a) Find AC, in terms of $\cos x$.

(b) Given that the area of triangle ABC is $50 \cos x$, find angle $\hat{C}$.

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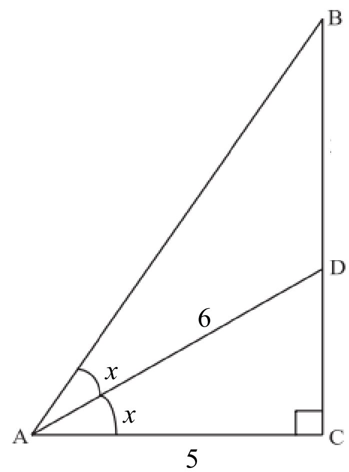
B. Exam style questions (LONG)

24. [Maximum mark: 13] ***[without GDC]***

Let ABC be the right-angled triangle, where $\hat{\mathrm{C}} = 90$. The line (AD) bisects $\mathrm{B\hat{A}C}$.

$\mathrm{AC} = 3$ and $\mathrm{AD} = 5$. as shown in the diagram.



(a) Write down the value of $\cos \mathrm{D\hat{A}C}$. [1]

(b) Find $\cos \mathrm{B\hat{A}C}$. [4]

(c) **Hence** find AB. [3]

(d) Find $\sin \mathrm{B}$. [2]

(d) Find $\tan \mathrm{BAD}$. [3]

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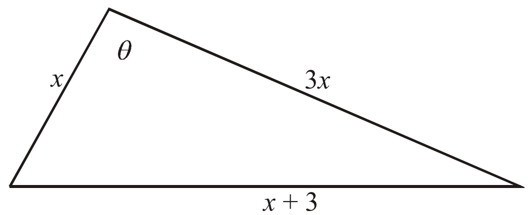
25*. [Maximum mark: 10] ***[with GDC]***

The area of the triangle shown below is $2.21\,\text{cm}^2$. The length of the shortest side is $x\,\text{cm}$ and the other two sides are $3x\,\text{cm}$ and $(x+3)\,\text{cm}$.



(a) Using the formula for the area of the triangle, write down an expression for $\sin\theta$ in terms of x. [2]

(b) Using the cosine rule, write down an expression for $\cos\theta$ in terms of x [2]

(c) (i) Using your answers to parts (a) and (b), show that,

$$\left(\frac{3x^2-2x-3}{2x^2}\right)^2 = 1-\left(\frac{4.42}{3x^2}\right)^2$$

(ii) Hence find

(a) the possible values of x;

(b) the corresponding values of θ, **in radians,** using your answer to part (b) above. [6]

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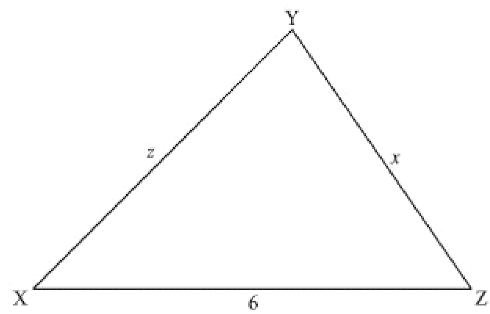
26*. [Maximum mark: 20] ***[with GDC]***

(a) Let $y = -16x^2 + 160x - 256$. Given that y has a maximum value, find

(i) the value of x giving the maximum value of y;

(ii) this maximum value of y.

The triangle XYZ has XZ = 6, YZ = x, XY = z as shown below. The perimeter of triangle XYZ is 16. [4]



(b) (i) Express z in terms of x.

(ii) Using the cosine rule, express z^2 in terms of x and $\cos Z$.

(iii) Hence, show that $\cos Z = \dfrac{5x-16}{3x}$. [7]

Let the area of triangle XYZ be A.

(c) Show that $A^2 = 9x^2 \sin^2 Z$. [2]

(d) Hence, show that $A^2 = -16x^2 + 160x - 256$. [4]

(e) (i) Hence, write down the maximum area for triangle XYZ.

(ii) What type of triangle is the triangle with maximum area? [3]

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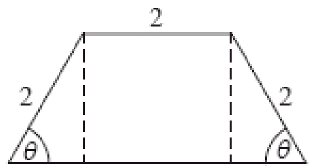
27. [Maximum mark: 10] ***[with GDC]***

The diagram below shows a plan for a window in the shape of a trapezium.



Three sides of the window are 2 m long. The angle between the sloping sides of the window and the base is θ, where $0 < \theta < \frac{\pi}{2}$.

(a) Show that the area of the window is given by $y = 4\sin\theta + 2\sin 2\theta$. [5]

(b) Zoe wants a window to have an area of 5 m^2. Find the two possible values of θ. [3]

(c) John wants two windows which have the same area A but different values of θ. Find all possible values for A. [2]

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