# NOTES ON ENERGY, WORK, MACHINES

## INTRODUCTION

***"The most important and most fruitful concepts are those to which it is impossible to attach a well-defined meaning."***

The concepts of energy and work are abstractions, are not fundamental laws of nature, and are not even part of Newton's *Principia*. Indeed, these ideas were argued about for some years before agreement was reached that they are both useful concepts. There is no obvious physical interpretation of work or energy, because they are human-created definitions. These definitions, however, can be developed using Newton's laws of motion.

The so-called integrals of motion provide a mathematical basis for going from Newton's Second Law to the definitions of impulse and work. If we integrate a force, which need not be constant, across a time interval, we get the impulse, and this can be shown to cause a change in the momentum of the object upon which the force is acting.

But we can also integrate the force across the *distance* over which it acts. This quantity is defined to be the work done by the force, and it causes a change in something that we will define to be the "kinetic energy" of the object. Note that, when the force in question is constant over the interval (time or distance) of interest, the calculus process of integration reduces to simple multiplication.

The difference between integrating across time as opposed to across distance is not, to say the least, intuitively obvious. However, the concepts of work and energy are, without question, very useful in physics. Some problems that would be very difficult and tedious to analyze, if a solution could be found at all, using just Newton's laws and kinematics become elegantly simple if we use work and energy instead.

## ENERGY

Although we use the word "energy" frequently, it is not easy to define in physics. One idea is that energy is the ability of an object to change the state, or condition, of another object. Another definition is that "energy is the capacity for doing work." Many physical processes represent a conversion or transformation from one form of energy to another; energy can also be transferred from one object to another. Energy is a scalar quantity; it does not have a direction in the sense that a vector does.

There are two main kinds of energy in mechanics: **potential** and **kinetic**, and the sum of these is the **mechanical energy** of a system. Potential energy can be thought of as the energy of position for an object in a system. That object has the *potential* for doing work. Kinetic energy is the energy of motion.

## POTENTIAL ENERGY

The most common form of potential energy in mechanics (as opposed to, say, electromagnetics) is *gravitational* potential energy. It can be proved, using Newton's Law of Gravitation and some calculus, that the gravitational potential energy for an object near the earth’s surface is

$$E_P = m\,g\,y \qquad (1)$$

where $y$ is the vertical height above some reference, or zero, level. *This reference level is arbitrary*; in many cases we take it to be the ground (earth's surface), but it need not be. More correctly, to acknowledge that potential energy is relative, and that as a practical matter we are usually interested in a **change** in potential energy, we can write

$$\Delta E_P = m\,g\,\Delta y \qquad (2)$$

When $y$ is measured with respect to some given reference level, that level cancels out of the difference $\Delta y$-- it vanishes. Notice that the change in potential energy will be negative when an object falls, since its initial $y$ will be larger than the final $y$, so that

$$\Delta E_P = E_{P,\,final} - E_{P,initial} = m\,g\left(y_{final} - y_{initial}\right)$$

It can also be proved that the gravitational potential energy of an object *does not depend on the path taken* to get it to the relative height $y$. If we lift a box of books onto a tabletop, it has a certain amount of potential energy, given by Eq(1), regardless of whether we (a) lifted the box directly up; (b) used a ramp to push it up; or (c) walked all over the building, up and down stairs, before placing the box on the table.

## KINETIC ENERGY

Kinetic energy is much simpler to grasp-- it is the energy of motion and is *defined* as

$$E_K \equiv \frac{1}{2}\,m v^2 \qquad (3)$$

This definition arose from the integral of motion that uses distance-- the "work" integral (next section). Long ago, a lot of argument among scientists took place about whether this quantity or momentum ($\mathbf{p} = m\mathbf{v}$) was a better representation of the "quantity of motion." Recall that momentum is a vector, while all forms of energy are scalars.

## WORK

In very general terms, that also apply in other areas of physics besides mechanics, work can be thought of as the process of energy transfer. (But, if we have defined energy as the capacity to do work, then our definitions are circular!) When a net force acts on an object, over some distance, we say that the force does "work" on the object.

A force which only changes the direction of motion, while still causing an acceleration, does not alter the kinetic energy of the object. Such a force does not do work, but it does cause a change in the momentum of the object (which is a vector). Work is only done when motion is produced by the application of a force.

In general, if a constant net force **F** is acting at an angle $\theta$ (which may be zero) to the direction of object's displacement **d** (a vector), then the work done is found using

$$W = \mathbf{F}\bullet\mathbf{d} = F\,d\,\cos(\theta) = F_{\parallel}\,d \qquad (4)$$

If the force is not constant then we must use an integral in Eq(4). Work is the product of two vector quantities (force and displacement), but the result is a scalar; the vector part of Eq(4) (bold letters) is the so-called "dot" or "scalar" product of two vectors.

This is evaluated as indicated in Eq(4), using the product of the magnitudes of the two vectors, times the cosine of the angle between them. The last term in Eq(4) is another way to write this: the product of the displacement magnitude and the component of the force that is parallel to the displacement.

The cosine accounts for the fact that only the component of the force which acts along the direction of motion will cause a change in speed and, thus, kinetic energy. A more general treatment of work requires the use of vector calculus (a varying force, in two or three dimensions).

Neither work nor energy has a direction, but work can have an algebraic sign. Note that, if the direction of the force is opposite that of the motion, so that the angle $\theta$ is 180 degrees, then the work is negative. This is the case, for example, in stopping a moving car; the brakes do negative work.

To summarize a few properties of work: (a) it is a scalar, even though it is found from vector quantities; (b) it can be either positive or negative; (c) if the angle $\theta$ is 90 degrees, no work is done, since the applied force is only changing the direction of motion and not the magnitude of the object's velocity; (d) a common way of defining work is to say that it is "force times distance" but we see from Eq(4) that it may be a bit more complicated than that; (e) **the units of work are "joules" which are also the units of energy**. A joule is one newton-meter, or

$$1 \text{ joule } = 1 \text{ newton} \bullet \text{meter } = 1 \text{kg } \frac{\text{m}^2}{\text{s}^2}$$

The work done to lift an object of mass *m* across a height difference $\Delta y$ at a constant speed is just the force (to overcome gravity) times the distance lifted; so

$$W = \Delta E_P = m\, g\, \Delta y \tag{5}$$

In this case the work done in lifting is "stored" as potential energy. Note that, even though we lift straight up, the force is applied in the direction of the (vertical) displacement, so that the angle $\theta$ in Eq(4) is zero.

Also, when work is done on an object and its potential energy is not changed, we can write

$$W = \Delta E_k = \frac{1}{2}\, m \left( v_f^2 - v_i^2 \right) \tag{6}$$

This says that the work done by a net force acting on an object causes a change in the kinetic energy of the object. This equation is usually what is meant by the phrase "**work-energy theorem**." This result can be obtained via calculus (and *must* be, if the applied force is not constant), or by combining $F = m\, a$ and kinematics equations, for a constant force.

## CONSERVATION OF (MECHANICAL) ENERGY

A very important concept in physics is the "conservation of energy" (CoE) and in the present context this is *mechanical* energy. The conservation of energy says that, if no external force or "agent" acts on a system, *the sum of the potential and kinetic energies of the system remains constant*. Energy exchanges back and forth in the system, between potential and kinetic, but their sum is constant. Many mechanics problems, especially those involving velocities, are far more easily solved using the conservation of energy rather than the kinematics equations.

This concept is generalized into the oft-quoted statement: "Energy can neither be created nor destroyed." This implies that all the energy in the universe was present from the beginning and always will be. It can change forms, including into and back from matter, that is, by $E = mc^2$, but energy is conserved. (There are examples in nuclear science where this exchange between matter and energy is observable.)

A more mundane example of CoE is the simple pendulum. The potential energy varies with the elevation (height) of the bob, increasing as it goes higher, decreasing to zero when it is at the vertical (rest) position. The kinetic energy is the opposite, since it is zero when the bob is at its highest point- it momentarily stops, then reverses direction. As the bob passes through the vertical position it has the maximum speed. The sum of the mechanical energy, potential plus kinetic, is a constant. This constant is determined by both the initial velocity and angle at which the pendulum was released.

## GENERAL WORK-ENERGY THEOREM

The conservation of energy leads to this important result. When work is done on a system by an external agent, the "work-energy theorem" says that

$$W = \Delta E_P + \Delta E_K = m\, g \left( y_f - y_i \right) + \frac{1}{2} m \left( v_f^2 - v_i^2 \right) \tag{7}$$

How much of the work goes into kinetic vs. potential energy change depends on the specific problem. For one example, if there is no change in (gravitational) potential energy, then we can see from Eq(7) that the kinetic energy would be equal to the work done in bringing an object to the speed $v_f$ from rest. For another example, if we lift an object straight up at a constant speed then the last term in Eq(7) is zero, and we have Eq(5) for the work done in lifting an object. Note that none of the applied force goes into changing the speed of the object as we lift it.

If there is no external force applied to a system, and there are no "dissipative" or "nonconservative" losses (i.e., a loss of energy that cannot be recovered, such as frictional losses as heat), then Eq(7) leads to the important result

$$0 = \Delta E_P + \Delta E_K \quad \Rightarrow \quad \Delta E_K = -\Delta E_P \tag{8}$$

and this in turn is usually rearranged to be a very useful expression of the CoE:

$$\frac{1}{2} m v_i^2 + m g y_i = \frac{1}{2} m v_f^2 + m g y_f \tag{9}$$

## POWER

When work is done, it is over some finite time period. How rapidly work is done is measured by “power”

$$\bar{P} = \frac{W}{\Delta t} \tag{10}$$

This equation gives the average power produced (or used) over a time **interval**; thus the use of $\Delta t$, not the continuous time variable $t$. If the work done (or energy transferred) was not at a constant rate, then Eq(10) would need to use an integral. The units of power are “watts” which are defined to be

$$1 \, \frac{\text{joule}}{\text{s}} = 1 \, \text{watt}$$

Work, or energy, can be expressed using the product of power and time, so that

$$W = \Delta E = \int_{t_1}^{t_2} P(t)\, dt = \bar{P} \, \Delta t \tag{11}$$

A familiar unit is the “kilowatt-hour” that is used to measure electricity consumption. It measures *energy*, not power, and is done using a device (a KWH meter) that in effect calculates the integral in Eq(11), since the power usage (i.e., rate of energy consumption) in a home is not constant.

We also write here an equation that can be useful for some problems:

$$P = \frac{F\, d \cos(\theta)}{\Delta t} = \mathbf{F} \bullet \mathbf{v} = F \cos(\theta)\, v \tag{12}$$

since the displacement $d$ is like a $\Delta x$, and $\Delta x / \Delta t$ is just the velocity $v$. Note that in this case the angle $\theta$ is between the applied force and velocity vectors.

## SIMPLE MACHINES

A machine is a device that "multiplies" forces or changes the direction of forces. Usual examples are the lever, or a ramp (inclined plane), which everyone has used at some time. The basic idea is that the work input is equal to the work output (by conservation of energy). This leads to a simple relation that we can use to solve many of these kinds of problems:

$$W_{input} = F_{input}\ d_{input} = F_{output}\ d_{output} = W_{output} \tag{13}$$

Here $d$ is the distance moved. What this amounts to, for the lever especially, is that we can use less force to move something, but the price for this is that the object doesn't move very far. The relationship between the distances moved (*d*) and the lengths (*r*) of the lever on either side of the fulcrum (Type I lever, like a see-saw) can be shown, with a bit of math, to be

$$\frac{d_{output}}{d_{input}} = \frac{r_{output}}{r_{input}}$$

## MECHANICAL ADVANTAGE; EFFICIENCY

Mechanical advantage (MA) is a concept that expresses how a machine reduces the force we need to apply in order to do a certain amount of work. The larger MA is, the smaller force we need. The "ideal mechanical advantage" is usually defined as:

$$IMA \equiv \frac{d_{input}}{d_{output}} \tag{14}$$

while the "actual mechanical advantage" is defined to be

$$AMA \equiv \frac{F_{output}}{F_{input}} \tag{15}$$

and the "efficiency" is just the ratio of these:

$$\varepsilon \equiv \frac{AMA}{IMA} = \frac{F_{output}\, d_{output}}{F_{input}\, d_{input}} = \frac{W_{output}}{W_{input}} \tag{16}$$

The difference in actual vs. ideal MA is of course due to dissipative losses such as friction. Thus, the efficiency is less than unity.

An interesting fact is that the work done in using a frictionless ramp (efficiency = 1) to slide an object up to some given height is exactly the same as it would be if we just lifted the object directly up. This can be proved in various ways. What the ramp does for us is to reduce the applied force needed. If there is friction, however, the problem is a bit more subtle, and we could actually end up doing *more* work using the ramp than by direct lifting.

## EQUATION SUMMARY

$\Delta E_P = m\,g\,\Delta y$ **change in gravitational potential energy**

$E_K \equiv \frac{1}{2}\,m v^2$ **kinetic energy**

$W \;=\; F\,d\,\cos(\theta) \;=\; F_{\parallel}\,d$ **work (constant force)**

$W = \Delta E_P \;+\; \Delta E_K \;=\; m\,g\,\left(y_f - y_i\right) \;+\; \frac{1}{2} m\left(v_f^2 \;-\; v_i^2\right)$ **general work-energy theorem**

$\Delta E_K = -\,\Delta E_P$ **conservation of mechanical energy**

$\frac{1}{2} m v_i^2 \;+\; m\,g\,y_i \;=\; \frac{1}{2} m v_f^2 \;+\; m\,g\,y_f$ **conservation of mechanical energy**

$\overline{P} = \frac{W}{\Delta t}$ **power as work over a time interval**

$P = F\cos(\theta)\;v$ **power as force with velocity**

$W_{input} = F_{input}\;d_{input} \;=\; F_{output}\;d_{output} \;= W_{output}$ **simple machine**

$IMA \;\equiv\; \frac{d_{input}}{d_{output}}$ **ideal mechanical adva**ntage

$AMA \equiv \frac{F_{output}}{F_{input}}$ **actual mechanical advantage**

$\varepsilon \;\equiv\; \frac{AMA}{IMA} \;=\; \frac{F_{output}\,d_{output}}{F_{input}\,d_{input}} = \frac{W_{output}}{W_{input}}$ **efficiency**