Polygons I

About this Lesson

This lesson explores characteristics of regular polygons. Students use a graphing calculator in parametric mode to graph the functions $x = \cos t,\ y = \sin t$ with a variety of t-steps to generate the polygons. Students then work collaboratively to determine how quantities such as the measures of interior, exterior, and central angles, the number of diagonals, and the perimeter and area of the polygon are related to the number of sides of the polygon.

Prior to the lesson, students should be able to calculate perimeter, circumference, and area of polygons and circles and should have experience with special right triangle ratios.

This lesson is included in Module 8 – Limits.

Objectives

Students will

- graph polygons using a graphing calculator in parametric mode.
- determine angle measures (interior, exterior, and central) in regular polygons.
- draw and count the diagonals (from one vertex and from all vertices) in regular polygons.
- calculate perimeter and area of regular polygons, using the measure of the central angle.
- develop formulas in regular polygons for the number of vertices, the number of triangles formed when one vertex is connected to each of the other vertices, the sum of the interior angle measures, the measure of each interior angle, the measure of each exterior angle, the sum of the measures of one interior and one exterior angle, the number of diagonals, the central angle measure, the perimeter if the radius is 1, and the area if the radius is 1.

Level

Geometry

Common Core State Standards for Mathematical Content

This lesson addresses the following Common Core State Standards for Mathematical Content. The lesson requires that students recall and apply each of these standards rather than providing the initial introduction to the specific skill. The star symbol (★) at the end of a specific standard indicates that the high school standard is connected to modeling.

Explicitly addressed in this lesson

Code	Standard	Level of Thinking	Depth of Knowledge
G-CO.13 (LTF extends to parametric mode & to other regular polygons)	Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.	Analyze	III

TEACHER

Code	Standard	Level of Thinking	Depth of Knowledge
F-BF.1a	Write a function that describes a relationship between two quantities. Determine an explicit expression, a recursive process, or steps for calculation from a context.★	Analyze	III
F-IF.3	Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ for $n \geq 1$ (n is greater than or equal to 1).	Analyze	III
F-TF.3	(+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.	Analyze	III
G-SRT.11	(+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).	Analyze	III
G-SRT.9	(+) Derive the formula $A = (1/2)ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.	Analyze	III

Common Core State Standards for Mathematical Practice
These standards describe a variety of instructional practices based on processes and proficiencies that are critical for mathematics instruction. LTF incorporates these important processes and proficiencies to help students develop knowledge and understanding and to assist them in making important connections across grade levels. This lesson allows teachers to address the following Common Core State Standards for Mathematical Practice.

Implicitly addressed in this lesson

Code	Standard
1	Make sense of problems and persevere in solving them.
2	Reason abstractly and quantitatively.
4	Model with mathematics.
5	Use appropriate tools strategically.
6	Attend to precision.
7	Look for and make use of structure.
8	Look for and express regularity in repeated reasoning.

LTF Content Progression Chart

In the spirit of LTF's goal to connect mathematics across grade levels, the Content Progression Chart demonstrates how specific skills build and develop from sixth grade through pre-calculus. Each column, under a grade level or course heading, lists the concepts and skills that students in that grade or course should master. Each row illustrates how a specific skill is developed as students advance through their mathematics courses.

6th Grade Skills/Objectives	7th Grade Skills/Objectives	Algebra 1 Skills/Objectives	Geometry Skills/Objectives	Algebra 2 Skills/Objectives	Pre-Calculus Skills/Objectives
Investigate limits using patterns, diagrams, geometric figures, tables, and/or graphs. (200_06.LI_H.01)	Investigate limits using patterns, diagrams, geometric figures, tables, and/or graphs. (200_07.LI_H.01)	Investigate limits using patterns, diagrams, geometric figures, tables, and/or graphs. (200_A1.LI_H.01)	Investigate limits using patterns, diagrams, geometric figures, tables, and/or graphs. (200_GE.LI_H.01)	Investigate limits using patterns, diagrams, geometric figures, tables, and/or graphs. (200_A2.LI_H.01)	Investigate limits using patterns, diagrams, geometric figures, tables, and/or graphs. (200_PC.LI_H.01)
Determine the measures of angles, perimeter, and/or area of a regular polygon inscribed in a circle as they relate to limits. (200_06.LI_H.04)	Determine the measures of angles, perimeter, and/or area of a regular polygon inscribed in a circle as they relate to limits. (200_07.LI_H.04)	Determine the measures of angles, perimeter, and/or area of a regular polygon inscribed in a circle as they relate to limits. (200_A1.LI_H.04)	Determine the measures of angles, perimeter, and/or area of a regular polygon inscribed in a circle as they relate to limits. (200_GE.LI_H.04)	Determine the measures of angles, perimeter, and/or area of a regular polygon inscribed in a circle as they relate to limits. (200_A2.LI_H.04)	Determine the measures of angles, perimeter, and/or area of a regular polygon inscribed in a circle as they relate to limits. (200_PC.LI_H.04)
			Develop the formulas for the measures of angles, perimeter, and/or area of a regular polygon inscribed in a circle by recognizing the patterns. (200_GE.LI_H.04)	Develop the formulas for the measures of angles, perimeter, and/or area of a regular polygon inscribed in a circle by recognizing the patterns. (200_A2.LI_H.04)	Develop the formulas for the measures of angles, perimeter, and/or area of a regular polygon inscribed in a circle by recognizing the patterns. (200_PC.LI_H.04)

Connection to AP*

AP Calculus Topic: Limits

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Materials and Resources

- Student Activity pages
- Graphing calculators with parametric mode capabilities
- Applet to see Archimedes' approach to the relationship among π, the area, and the circumference of a circle using the concept of limits of inscribed polygons
  http://www.math.psu.edu/courses/maserick/circle/circleapplet.html

Assessments

The following types of formative assessments are embedded in this lesson:

- Students engage in independent practice.
- Student cooperative learning groups present the calculations for their particular polygon.

The following additional assessments are located on the LTF website:

- Limits – Geometry Free Response Questions
- Limits – Geometry Multiple Choice Questions

Teaching Suggestions

Polygons I and Polygons II are best presented in succession since Polygons II depends on the information the students accumulate in Polygons I.

The perimeter and area questions may be included in a later unit of study.

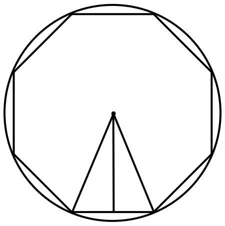
In Polygons I, lead the students through questions 1 – 9. On question 10, help the students complete the chart for a polygon with 5 and 7 sides (*t*-step 72 and $\frac{360}{7}$). Assign the other eight *t*-steps to small groups of students and ask them to enter their findings on a classroom copy of the summary table found at the end of the lesson. Students may need assistance with the development of some of the *n*-gon formulas after they have completed the chart for polygons with 3 to 12 sides. Lead the students through questions 11 – 16.

Help students understand the formula for the number of diagonals that can be drawn in a polygon. Explain that from a given vertex, diagonals cannot be drawn to the adjacent vertices because the segments formed are sides of the polygon. Also, a diagonal cannot be drawn from a particular vertex to itself. This means that, from a given vertex in an *n*-gon, only $n-3$ diagonals can be drawn. Moving around the polygon, this procedure can be repeated at each of the n vertices. Now $n(n-3)$ diagonals can be drawn; however, this process draws each diagonal twice so $n(n-3)$ must be divided by 2 to calculate the actual number of unique diagonals in a polygon.

Notes on determining the perimeter:

The answers shown on the teacher pages list both of the following methods of calculation to allow the teacher to choose which is most effective with his/her own students.

The traditional method for determining the perimeter is based on right triangle trigonometry. In an octagon, the process would be as follows: Determine the base of each isosceles triangle using right triangle trigonometry then multiply by eight because there are eight isosceles triangles around the octagon.



$$\textit{Central angle measure} = 45^\circ$$
$$\textit{Half the central angle measure} = 22.5^\circ$$
$$\textit{Base length of the triangle} = 2\sin 22.5$$
$$P_{octagon} = 8(2\sin 22.5) = 16\sin 22.5$$

For more advanced students, answers with square roots instead of trigonometric functions can be calculated using the Law of Cosines. In an octagon, the process would be as follows: Determine the base of each isosceles triangle using Law of Cosines then multiply by eight because there are eight isosceles triangles around the octagon.

$$x^2 = 1^2 + 1^2 - 2(1)\cos 45 = 2 - 2\left(\frac{\sqrt{2}}{2}\right); x = \sqrt{2-\sqrt{2}}$$

$$P_{octagon} = 8\sqrt{2-\sqrt{2}}$$

TEACHER

Notes on determining the area:
The answers shown on the teacher pages list both of the following methods of calculation to allow the teacher to choose which is most effective with his/her own students.

One traditional method that can be used to determine the area of an octagon is as follows:

$$A_{\Delta} = (\sin 22.5)(\cos 22.5) = \frac{\sqrt{2}}{4} \approx 0.354 \qquad A_{octagon} = 8\sin 22.5 \cos 22.5 = 2\sqrt{2} \approx 2.828$$

A second traditional method that can be used to determine the area of an octagon is as follows:

$$A_{octagon} = \frac{1}{2}(apothem)(perimeter) = \frac{1}{2}\cos 22.5(16\sin 22.5) = 8\cos 22.5 \sin 22.5 = 2\sqrt{2}$$

An alternate method to determine the area of an octagon is as follows: Determine the area of each isosceles triangle using the sine formula then multiple by eight because there are eight isosceles triangles inside the octagon.

$$A_{\Delta} = \frac{1}{2}ab\ \sin C = \frac{1}{2}(1)(1)\sin 45 = \frac{1}{2}(1)(1)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4} \qquad A_{octagon} = 8\left(\frac{\sqrt{2}}{4}\right) = 2\sqrt{2}$$

Modality
LTF emphasizes using multiple representations to connect various approaches to a situation in order to increase student understanding. The lesson provides multiple strategies and models for using these representations to introduce, explore, and reinforce mathematical concepts and to enhance conceptual understanding.

P – Physical
V – Verbal
A – Analytical
N – Numerical
G – Graphical

Answers

1. The sum of the exterior angle measures of a triangle, one at each vertex, is $360°$.
2. The measure of each exterior angle of an equilateral triangle, one at each vertex, is $120°$.
3.

Degrees	Exact Coordinates	Approximate Coordinates
30	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	(0.866, 0.5)
45	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	(0.707, 0.707)
60	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	(0.5, 0.866)

4. $x = \cos\theta;\ y = \sin\theta$
5. See student page instructions.
6. Circle
7.

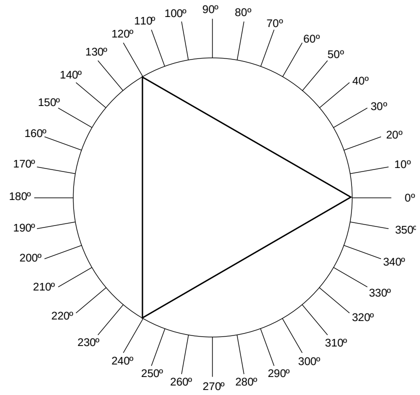
t-step	Approximate Coordinates
30	(0.866, 0.5)
45	(0.707, 0.707)
60	(0.5, 0.866)

8. Central angle measures
9. 360 degrees in a circle

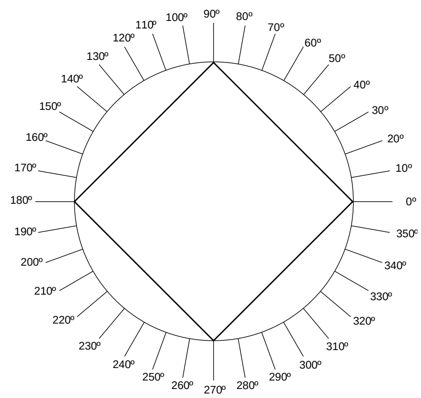
TEACHER

10. See graphs and summary chart.

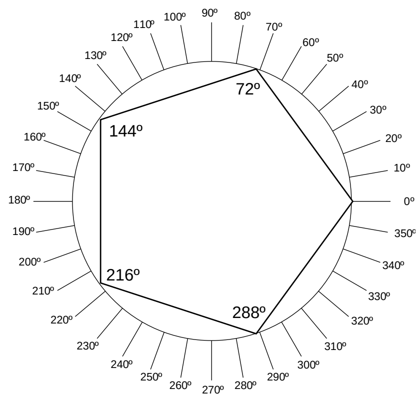
t-step: 120 polygon name: triangle



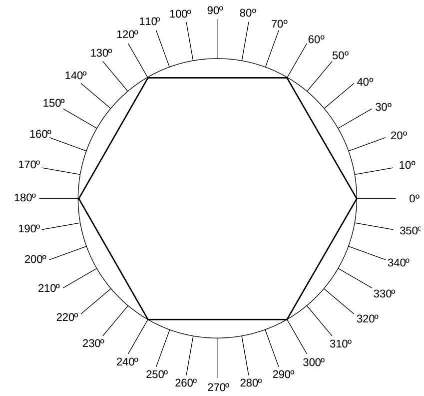
t-step: 90 polygon name: quadrilateral



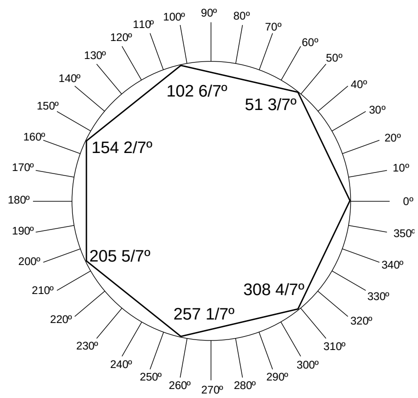
t-step: 72 polygon name: pentagon

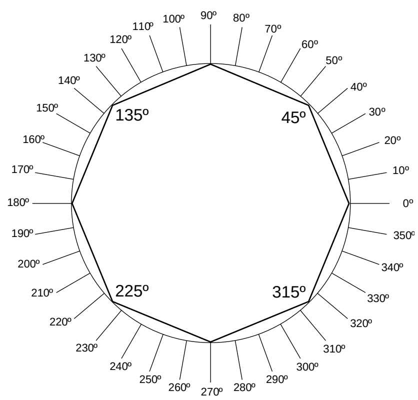


t-step: 60 polygon name: hexagon

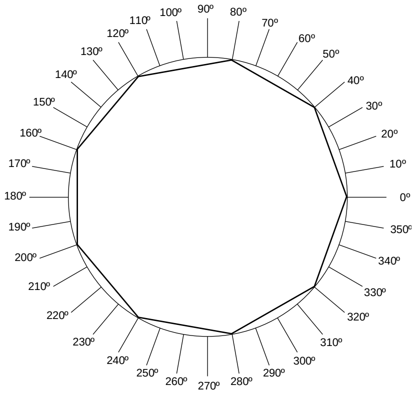


t-step: $\dfrac{360}{7}$ polygon name: septagon or heptagon

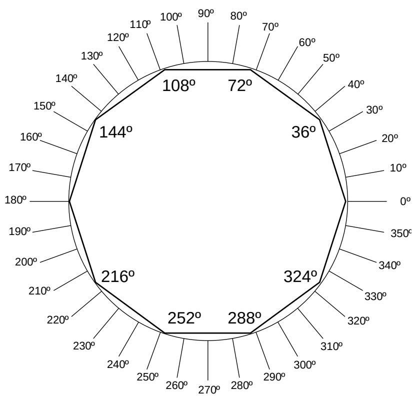




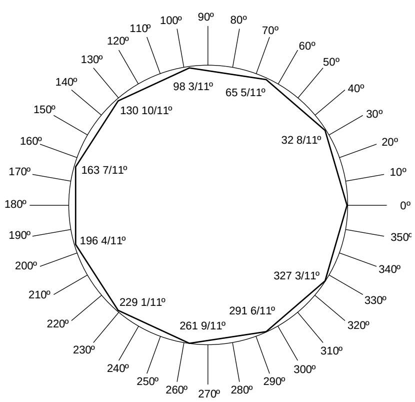
t-step: 40 polygon name: nonagon



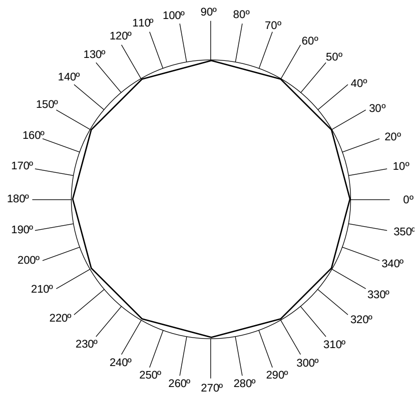
t-step: 36 polygon name: decagon



t-step: $\frac{360}{11}$ polygon name: undecagon



t-step: 30   polygon name: dodecagon



11. The number of sides increases.

12. The figure appears more circular.

13. 360/*t*-step = number of sides.

14. 36-gon; 360-gon
    So, in question #6, the graph with a *t* step of 10 appeared to be a circle. Actually, though, it was a 36-gon.

15. There is not a *t*-step that will graph a perfect circle. A *t*-step close to 0 will give a good approximation to a circle. (Note: On the graphing calculator, once the t-step is decreased to cover each pixel, the polygon will not appear any more circular.)

16. circle

TEACHER

# of sides	t-step	Name	# of Vertices	# of Triangles, one vertex connected	Sum of angle measures	If it is regular, measure of each interior angle	If it is regular, measure of each exterior angle	Sum of the measures of the exterior angles, one at each vertex	Number of diagonals	Central angle measure	Perimeter if radius is 1	Area if the radius is 1
3	120	triangle	3	1	180	60	120	360	0	120	$3\sqrt{3} \approx 5.196$	$\frac{3\sqrt{3}}{4} \approx 1.299$
4	90	quadrilateral	4	2	360	90	90	360	2	90	$4\sqrt{2} \approx 5.657$	2
5	72	pentagon	5	3	540	108	72	360	5	72	$10 \sin 36^\circ$ $= \frac{5\sqrt{10-2\sqrt{5}}}{2}$ ≈ 5.878	$5 \sin 36^\circ \cos 36^\circ =$ $\frac{5}{2} \sin 72^\circ \approx 2.378$
6	60	hexagon	6	4	720	120	60	360	9	60	6	$\frac{3\sqrt{3}}{2} \approx 2.598$
7	$\frac{360}{7}$	heptagon or septagon	7	5	900	$128\frac{4}{7}$	$51\frac{3}{7}$	360	14	$\frac{360}{7}$	$14 \sin\left(\frac{180}{7}\right)^\circ$ ≈ 6.074	$7 \sin\left(\frac{180}{7}\right)^\circ \cos\left(\frac{180}{7}\right)^\circ =$ $\frac{7}{2} \sin\left(\frac{360}{7}\right)^\circ \approx 2.736$
8	45	octagon	8	6	1080	135	45	360	20	45	$16 \sin 22.5^\circ$ $= 8\sqrt{2-\sqrt{2}}$ ≈ 6.123	$2\sqrt{2} \approx 2.828$
9	40	nonagon	9	7	1260	140	40	360	27	40	$18 \sin 20^\circ$ ≈ 6.156	$9 \sin 20^\circ \cos 20^\circ =$ $\frac{9}{2} \sin 40^\circ \approx 2.893$
10	36	decagon	10	8	1440	144	36	360	35	36	$20 \sin 18^\circ$ $= 5(\sqrt{5} - 1)$ ≈ 6.180	$10 \sin 18^\circ \cos 18^\circ =$ $5 \sin 36^\circ \approx 2.939$
11	$\frac{360}{11}$	undecagon	11	9	1620	$147\frac{3}{11}$	$32\frac{8}{11}$	360	44	$\frac{360}{11}$	$22 \sin\left(\frac{180}{11}\right)^\circ$ ≈ 6.198	$11 \sin\left(\frac{180}{11}\right)^\circ \cos\left(\frac{180}{11}\right)^\circ =$ $\frac{11}{2} \sin\left(\frac{360}{11}\right)^\circ \approx 2.974$
12	30	Dodecagon	12	10	1800	150	30	360	54	30	$24 \sin 15^\circ$ $=6(\sqrt{6}-\sqrt{2}$ ≈ 6.212	3
n	$\frac{360}{n}$	*n*-gon	*n*	*n*-2	180(*n*-2)	$\frac{180(n-2)}{n}$	$\frac{360}{n}$	360	$\frac{n(n-3)}{2}$	$\frac{360}{n}$	$2n \sin\left(\frac{180}{n}\right)^\circ =$ $n\sqrt{2-2\cos\left(\frac{360}{n}\right)^\circ}$	$n \sin\left(\frac{180}{n}\right)^\circ \cos\left(\frac{180}{n}\right)^\circ$ $= \frac{n}{2} \sin\left(\frac{360}{n}\right)^\circ$

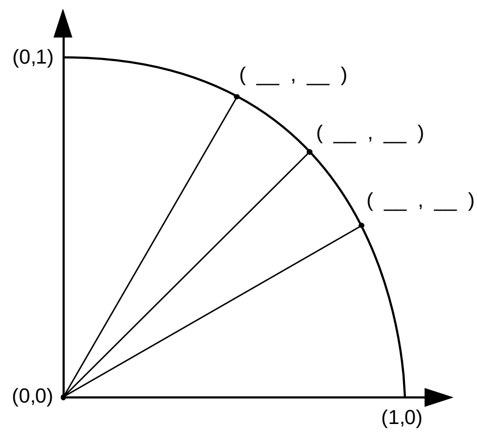
Polygons I

1. The sum of the exterior angle measures of a triangle, one at each vertex, is ________ .

2. The measure of each exterior angle of an equilateral triangle, one at each vertex, is ________ .

3. List the exact values, and then give the decimal approximation for each coordinate shown on the unit circle.

Degrees	Exact Coordinates	Approximate Coordinates
30		
45		
60		



4. Describe the coordinate points on the circle with trigonometric functions:

   x = __________ y = __________

5. Set your TI calculator as follows:

Mode	Window	Y=
Normal Sci Eng	Tmin=0	Plot1 Plot2 Plot3
Float 0123456789	Tmax=360	\X_{1T} = cos(T)
Radian Degree	Tstep=10	\Y_{1T} = sin(T)
Func Par Pol Seq	Xmin=-1.516	\X_{2T} =
Connected Dot	Xmax=1.516	\Y_{2T} =
Sequential Simul	Xscl=1	\X_{3T} =
Real a+bi re^θi	Ymin=-1	\Y_{3T} =
Full Horiz G-T	Ymax=1	\X_{4t} =
	Yscl=0	

6. Describe what *appears* to be shown on your graph. _______________ .

7. Use the trace key, type $t = 30$, and press enter to evaluate the following and list the (x, y) coordinates. Repeat for $t = 45$ and $t = 60$.

t-step	Approximate Coordinates
30	
45	
60	

8. What do the *t* values represent?

9. Why were the *t* values 0 to 360 chosen?

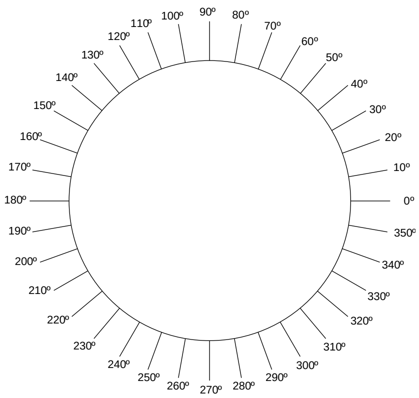
10. Using the pairs of circles provided on the following pages, complete the following.
    a. Change the *t*-step in the window of your calculator to the value shown above each graph.

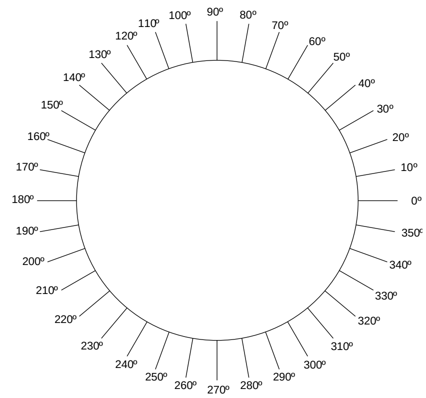
    b. Graph the polygon shown on your calculator on both of the circles provided.

    c. Use the first graph to complete the following information on the chart.
        i. Count the number of sides.
        ii. List the *t*-step.
        iii. Name the polygon.
        iv. Count the number of vertices of the polygon.
        v. Draw the diagonals connecting one vertex to each of the other vertices.
        vi. Calculate the sum of the angles of the triangles formed by connecting one vertex to the other vertices.
        vii. Determine the measure of each interior angle of this regular polygon.
        viii. Determine the measure of each exterior angle of this regular polygon.
        ix. Calculate the sum of the exterior angles of the polygon, one at each vertex.
        x. Draw and count all the diagonals of the polygon.

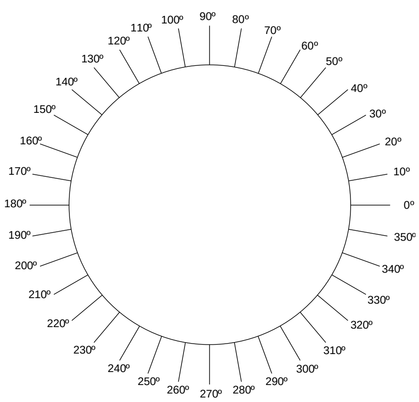
    d. Use the second graph to complete the following information on the chart
        i. Draw at least one central angle and determine its measure.
        ii. Calculate the perimeter of the polygon.
        iii. Calculate the area of the polygon.

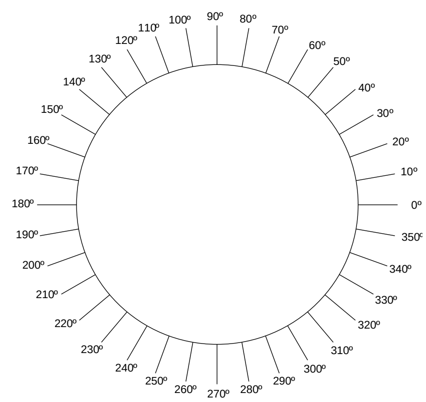
t-step: 120   polygon name: ____________________



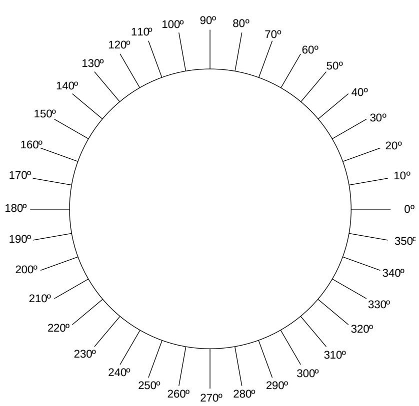


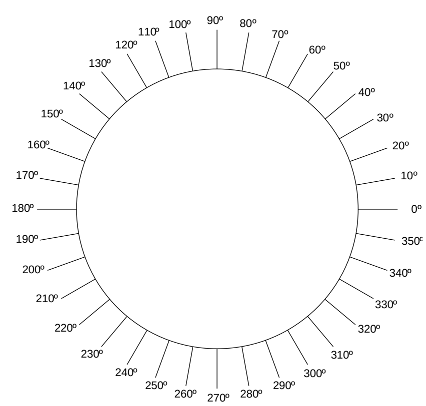
t-step: 90   polygon name: ____________________



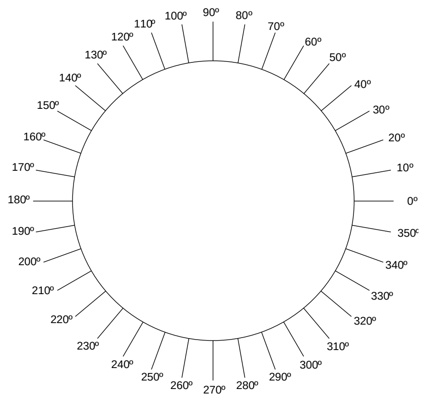


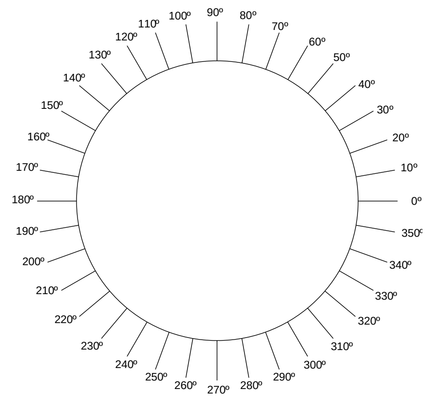
t-step: 72   polygon name: ____________________



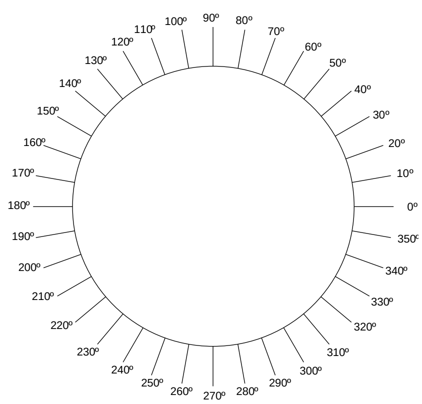


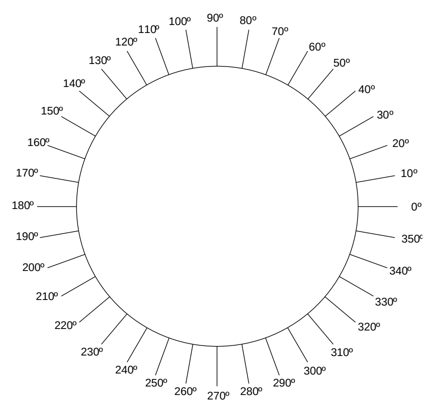
t-step: 60 polygon name: ____________________



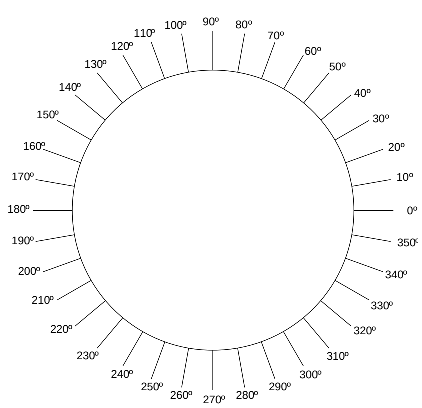


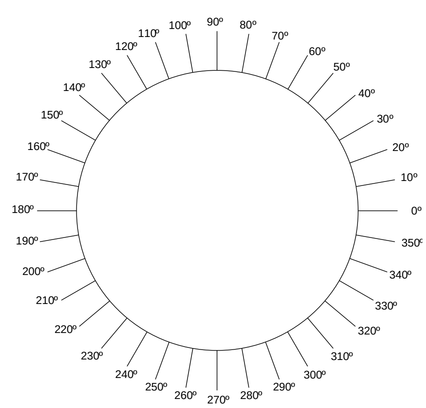
t-step: $\frac{360}{7}$ polygon name: ____________________



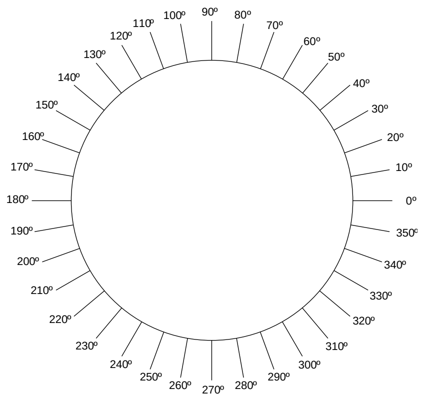


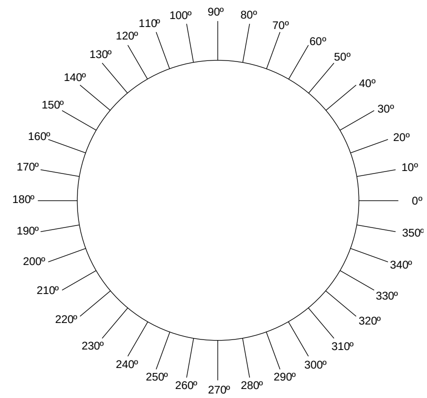
t-step: 45 polygon name: ____________________



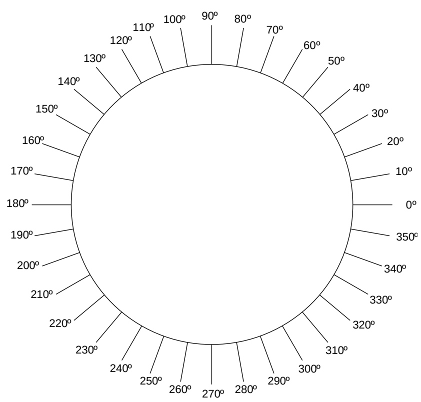


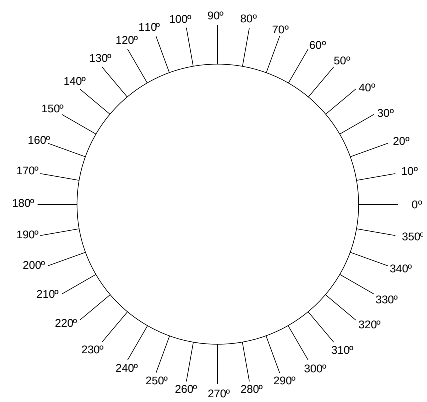
t-step: 40 polygon name: ____________________



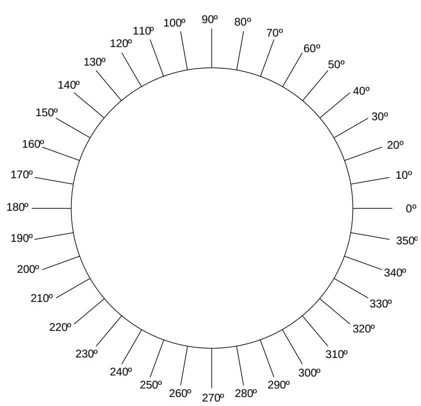


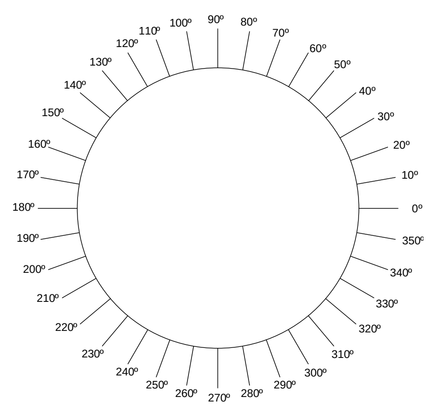
t-step: 36 polygon name: ____________________



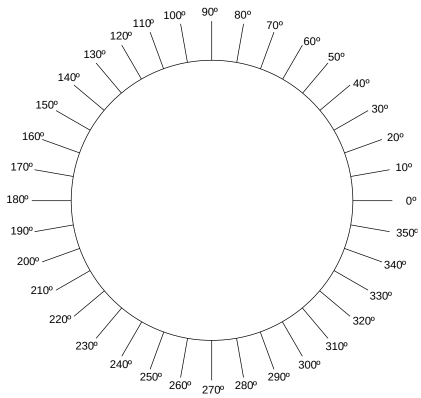


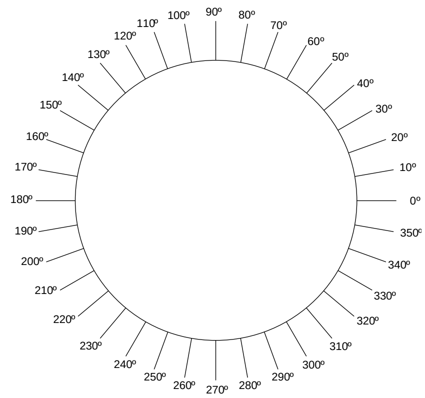
t-step: $\frac{360}{11}$ polygon name: ____________________





t-step: 30   polygon name: ____________________





Conclusions:

11. As the *t*-step decreases, what happens to the number of sides?

12. As the number of sides increases, the figure appears more ___________.

13. How is the *t*-step related to the number of sides?

14. Describe the figures graphed with the following:

    t-step = 10 _________________   *t*-step = 1 __________________

    So, in question 6, the graph with a *t*-step of 10 appeared to be a ______________ .

    Actually, though, it was a ____________________.

15. What would the *t*-step have to be to graph a perfect circle?
    (This is the calculus concept of a limit.)

16. As the number of sides increases, the area of the polygon becomes closer to the area of a ____________.
    (This is another example of the calculus concept of a limit.)

# of sides	*t*-step	Name	# of Vertices	# of Triangles, one vertex connected	Sum of angle measures	If it is regular, measure of each interior angle	If it is regular, measure of each exterior angle	Sum of the measures of the exterior angles	Number of diagonals	Central angle measure	Perimeter if radius is 1	Area if the radius is 1
	120											
	90											
	72											
	60											
	$\frac{360}{7}$											
	45											
	40											
	36											
	$\frac{360}{11}$											
	30											
	$\frac{360}{n}$											