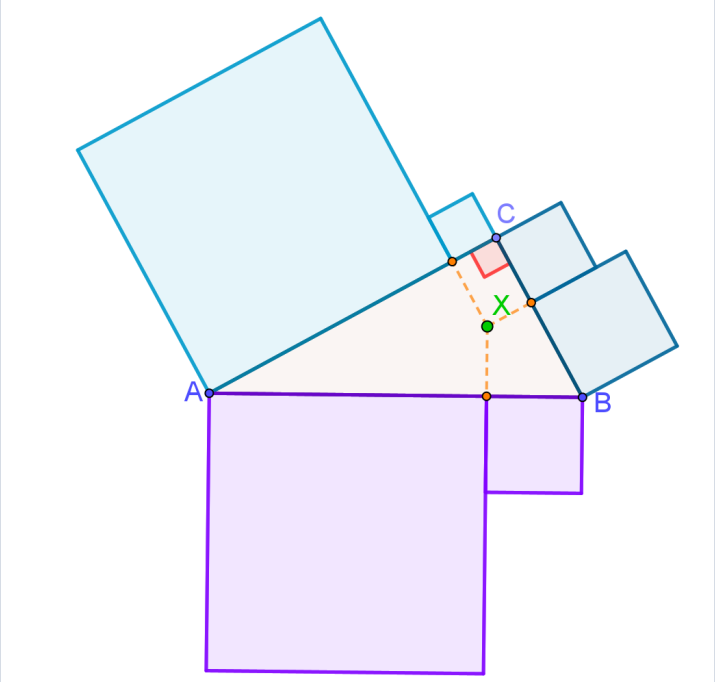
DYNAMIC VISUALIZATION

as a principle for teaching and learning mathematics with digital learning environments

Hans-Jürgen Elschenbroich, Korschenbroich



What is this talk – not?

- **No research questions, no empirical study!**
- **But: pedagogical experience in school, subject-matter didactics (*Stoffdidaktik*).**
- **My person: teacher, teacher education & trainer, media consultant.**

Conceptual Clarifications

- An **image** (*Abbild*) refers directly to an *object*.
- An **illustration** (*Veranschaulichung*) serves a *didactic function*:
  it is intended to make the subject matter (more) *comprehensible*.
- **Visualization** (*Visualisierung*) serves a more pronounced didactic function: it “refers to the process of translating complex relationships into visual media in order to make them *understandable*”. (Wikipedia)
  - **Static illustrations**: sketches, diagrams, and graphics have been common in the teaching and learning of mathematics for centuries.
  - An initial form of **dynamization** emerged with movable drawing instruments (pantograph, ellipse compass, integraph) and later with sequences of images and instructional films.

Dynamic visualization

- **Dynamic mathematics software** (DynaGeo, Cabri Géomètre, Geometer's Sketchpad, nowadays GeoGebra) significantly expands the didactic possibilities of traditional media. Mathematical objects and configurations change over time, ideally not as passive consumption of videos, but through an *interactive process* involving (virtual) actions by learners and the associated cognitive activities (Elschenbroich, 2004).
- **Dragging mode** with movable points and sliders enriches mathematics instruction as new objects and traces/loci become immediately accessible.
- **Dynamic visualization** has become an important instructional principle. But caution: Dynamic visualizations are intellectually demanding; there is *no didactic benefit at zero cost* (Hölzl, 1994).
- There is the risk of *cognitive overload* and excessive *acceleration*. It is advisable to combine it with action-oriented phases of deceleration (Heintz et al., 2017; Mayer & Moreno, 2003).

Systematic Variation

- Another instructional principle that is connected with the dissemination of digital mathematics tools and dynamic visualization is **Systematic Variation**.
- Learners often tend towards unsystematic and ‘wild’ dragging. However, greater dynamism of this kind does not produce greater insight; rather, it may hinder it.
- Crucial are *suitable guidance* and “the conscious, *goal-directed* modification of ‘influencing factors’ as well as the observation and interpretation of the resulting changes in the dependent variables” (Roth, 2008. Translation by the author)

Instructional Principles

Important *General* **Instructional Principles** for educational and instructional design:

- **Genetic principle** (Martin Wagenschein)
- **Operative principle** (Jean Piaget)
- **Spiral principle** (Jerome A. Bruner)
- **Principle of multiple representations** (Jerome A. Bruner)
  CPA approach (**c**oncrete, **p**ictorial, **a**bstract), **EIS** principle (**e**nactive, **i**conic, **s**ymbolic)

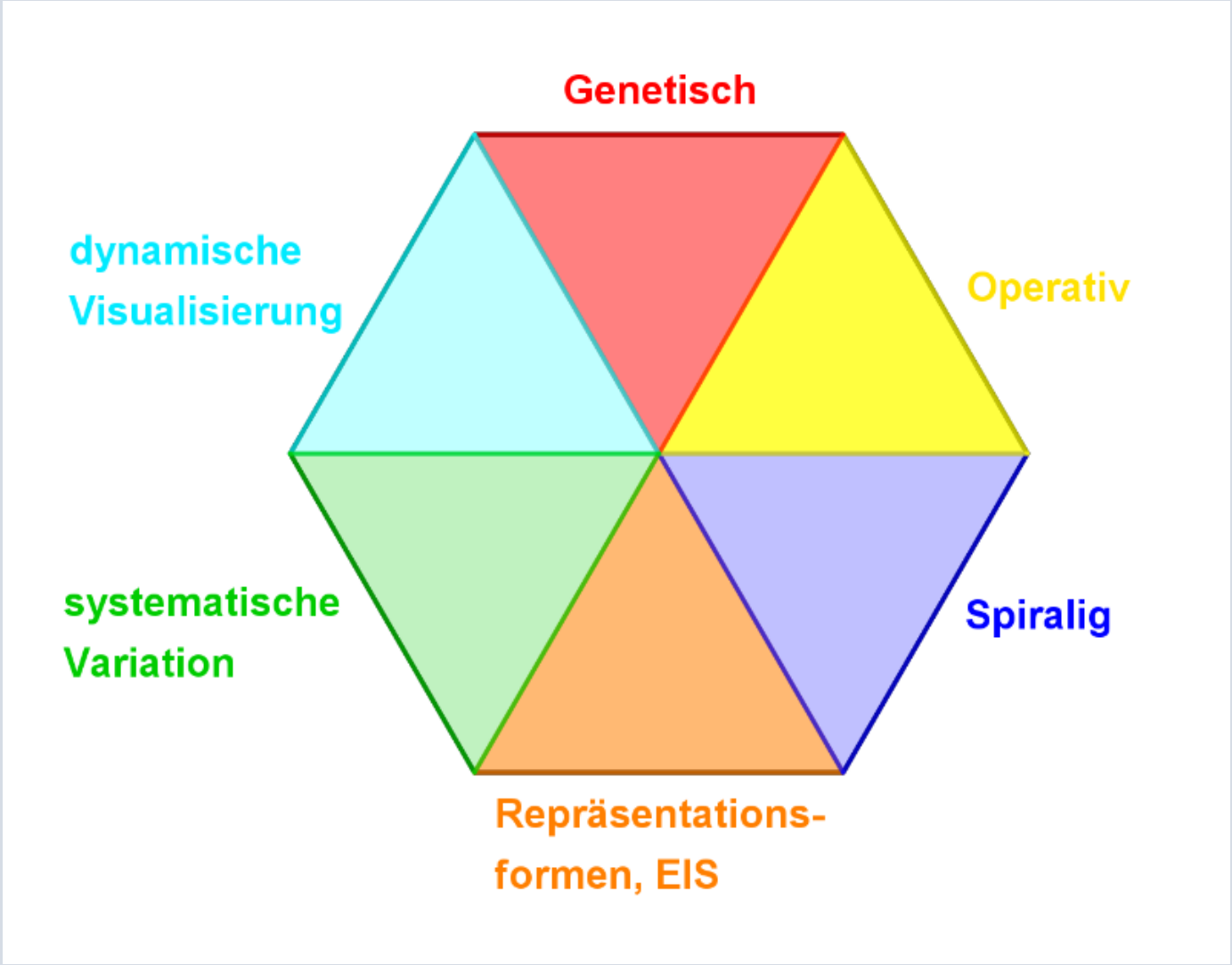
(Elschenbroich & Dutkowski, 2023; Heitzer & Weigand, 2020)

Specific Digital Instructional Principles:

- **Dynamic Visualization**
- **Systematic Variation**

(Elschenbroich & Dutkowski, 2023; Heintz et al, 2017).

Instructional Principles



Digital Mathematics Learning Environments

- In the early days of computer use, lessons always started with a blank screen, and all constructions were developed together in class.
- Nowadays, dynamic learning environments have become established in the form of digital mathematics learning environments (DMLE) (Weinhandl, Baldinger & Kapplmüller, 2025).
- DMLE emerged (in Germany) as ‚electronic worksheets‘ or ‚interactive worksheets‘ (EIWOS), made with *DynaGeo Euklid* or *Cabri II)* (Elschenbroich & Seebach, 1999; Elschenbroich, 2001).
- They consist of an informative text, a dynamic file (today usually created with GeoGebra), and guided-learning tasks, possibly including solutions for self-study control. All components are integrated into a web-based file.
- DMLE cannot be reduced to a mere 'artefact’, but must be understood as an instrument (Rabardel, 1995/2002; Guin & Trouche, 2002).
- Multimedia learning environments do not trigger the learning process itself automatically (Schnotz & Lowe, 2003)!

Use of digital media in teaching and learning settings

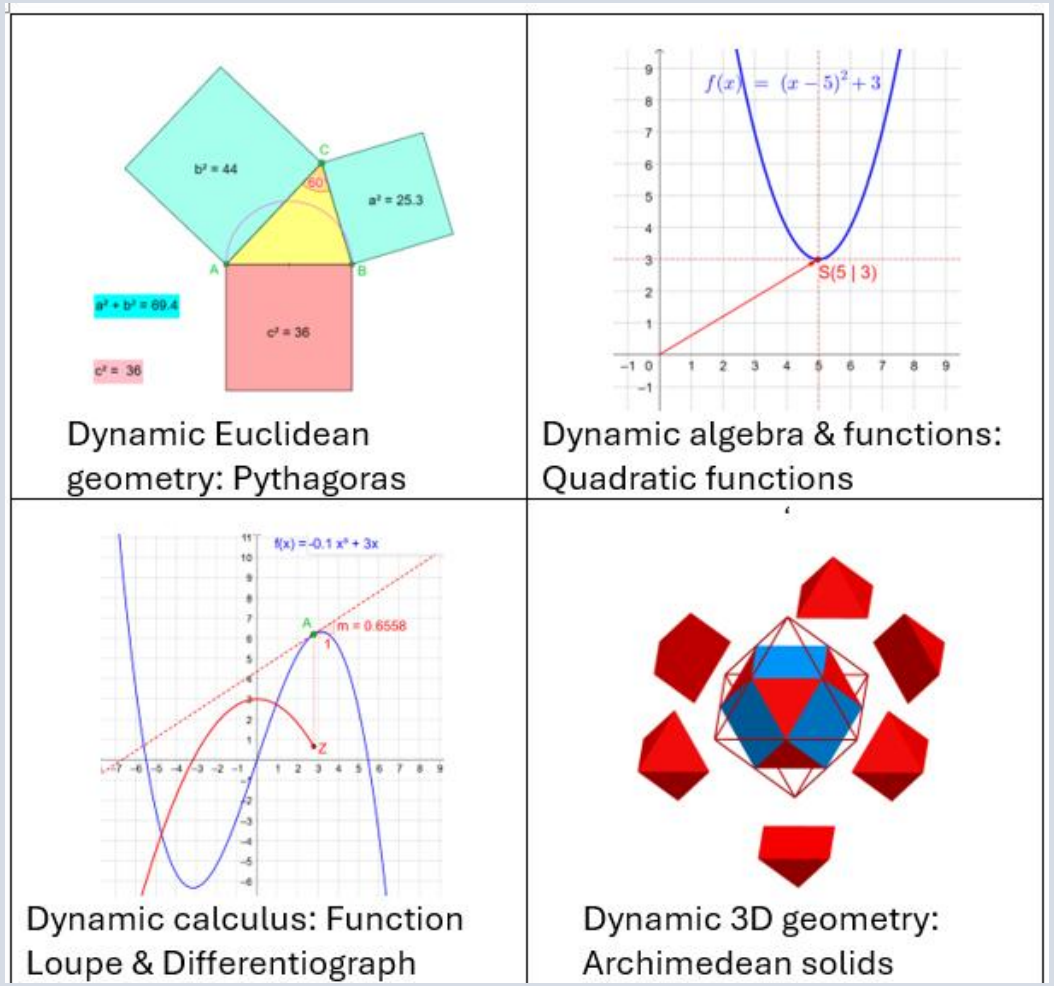


https://www.digillab.mcls.uni-muenchen.de/bilder/icap-ohne-ladebalken1.png

DMLE: interactiv.

Agenda *(Examples I wanted to talk about)*

Examples of dynamic mathematics and DMLE



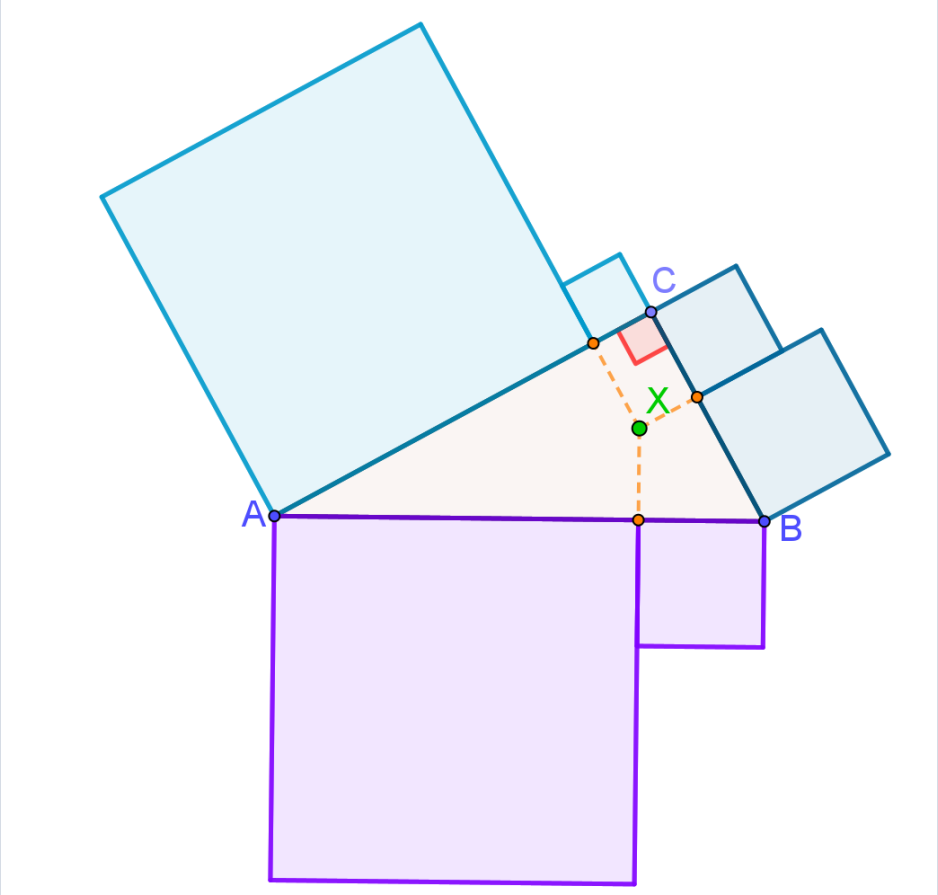
Agenda *(Example I will talk about)*

Examples of dynamic mathematics and DMLE:

1. **Bithagoras**, a biquadratic generalization of the Pythagoras Theorem.
2. ~~Dynamic Euclidean geometry: Pythagoras~~
3. ~~Dynamic algebra & functions: Quadratic functions & equations~~
4. ~~Dynamic calculus: Function Loupe & Differentiograph~~
5. ~~Dynamic 3D geometry: Archimedean solids~~

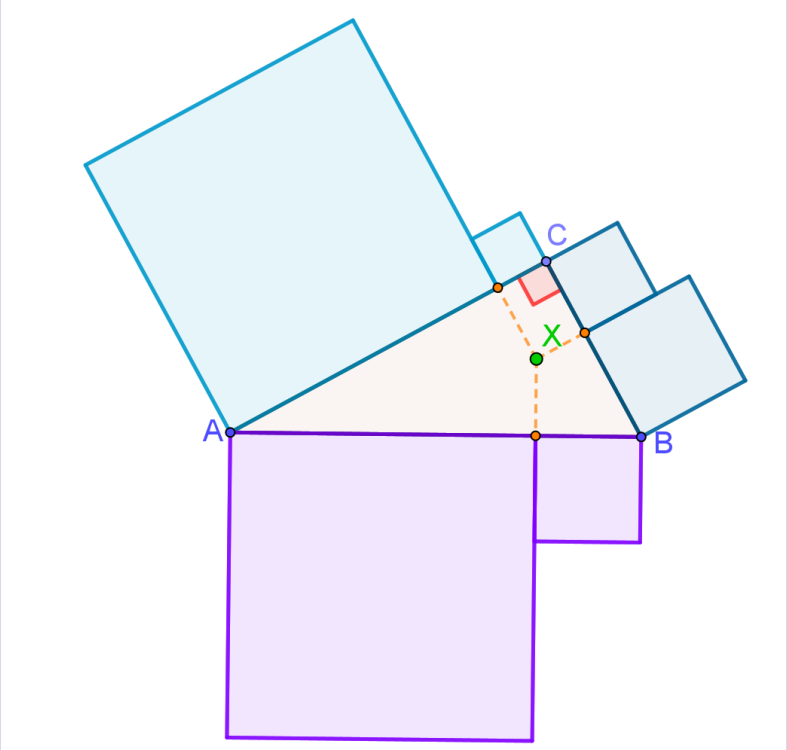
Example 1: Bithagoras. Problem posing

- “What happens to a right-angled triangle ABC if there are *two* squares on each side of the triangle, connected by the perpendicular feet of a point X?”

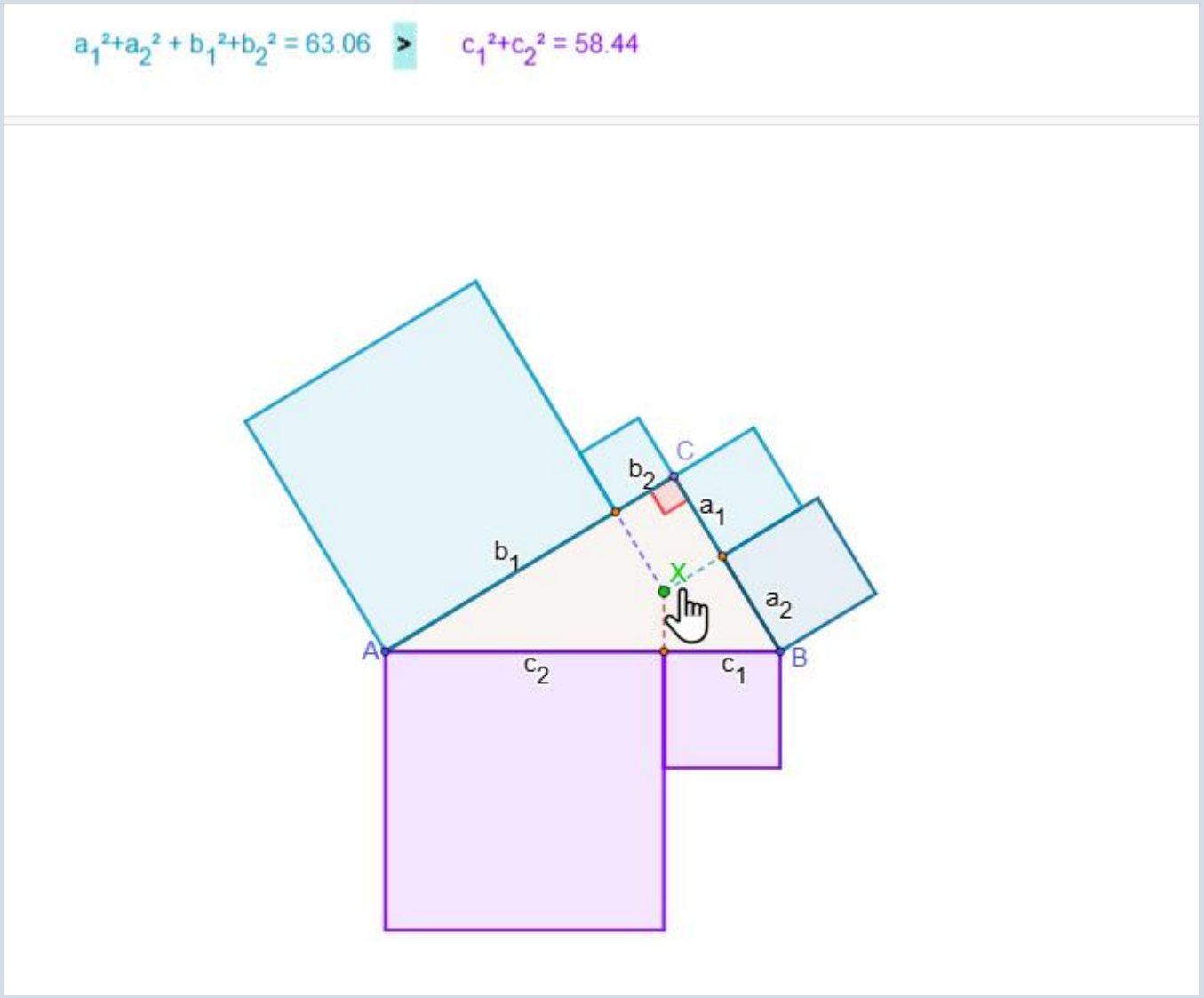


GeoGebra construction

- Right-angled triangle ABC.
  Point X is inside ABC and its perpendicular feet intersect a, b, c.
  Create squares and add their areas. Drag X.
- Is the sum of the blue squares on the legs a and b equal to the sum of the violet squares on the hypotenuse c? Or greater? Or smaller?



Empirical investigation by dynamic visualization





TQRCG

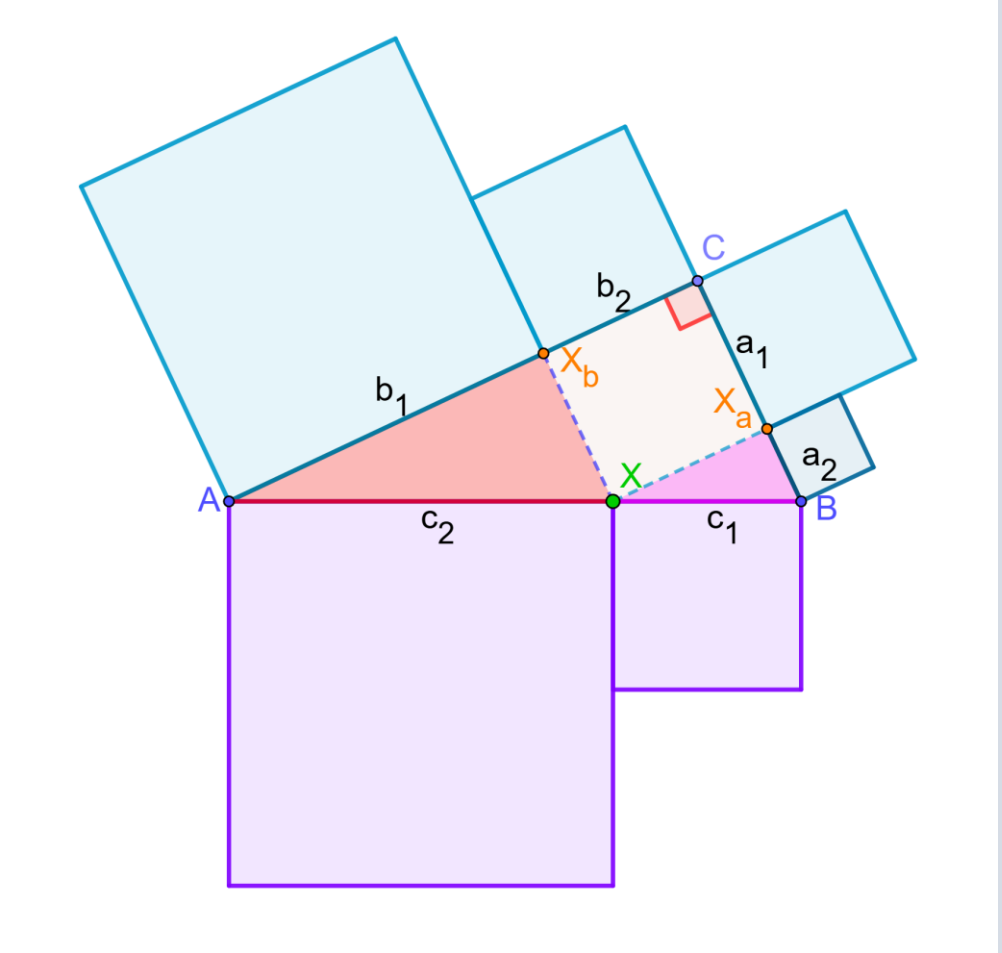
Bithagoras (Pythagoras Generalization)

- Dynamic visualization shows: $a_1^2 + b_1^2 + a_2^2 + b_2^2 > c_1^2 + c_2^2$, if X is *inside* ABC.
- For X *on* c = AB, equality holds.
- And if X = A or X = B, then we get the classical Pythagoras figure.

- So with Dynamic Visualization we have *detected* a relationship between the squares.
- But *WHY must* it be so?

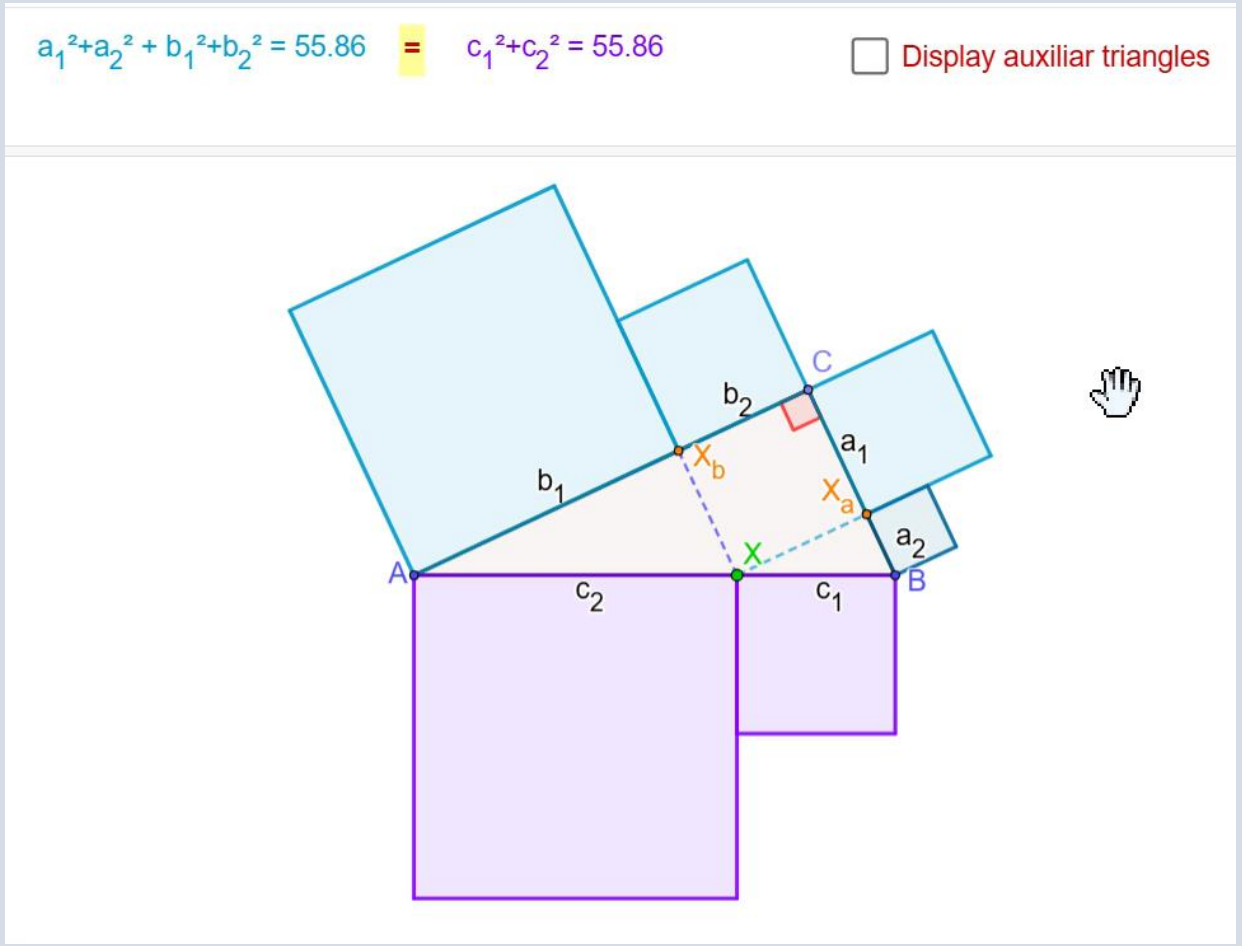
Proof: Start with a special case

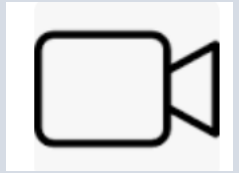
- Let be X on c = AB.
- Then we have two right-angled triangles AXX_b (red) and XBX_a (magenta).
- It is obvious that $a_1^2 + b_1^2 = c_2^2$ and $a_2^2 + b_2^2 = c_1^2$.
- So $a_1^2 + a_2^2 + b_1^2 + b_2^2 = c_1^2 + c_2^2$.



Mathematical proof by systematic variation

- X on c = AB. Now we move X only perpendicular to c (i.e. c_1 and c_2 are fixed).

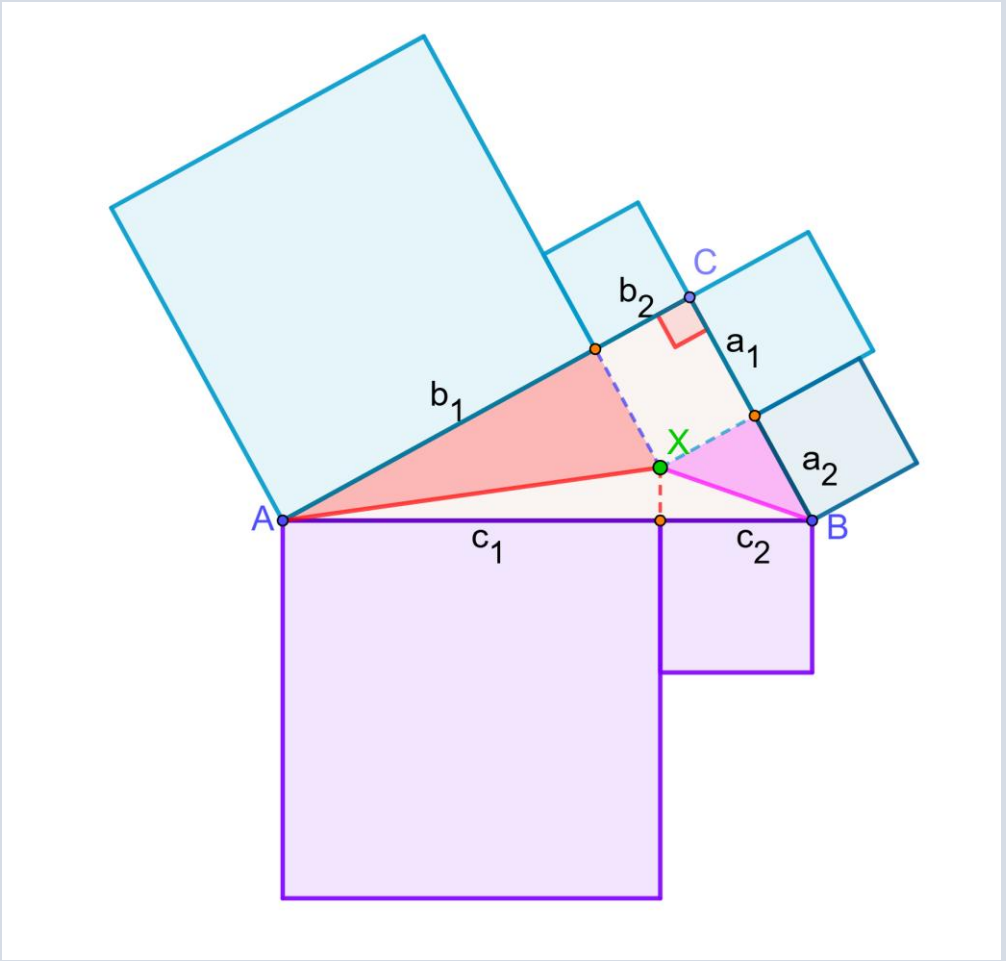






Bithagoras (Pythagoras Generalization)

- X on c = AB. We move X only perpendicular to c (i.e. c_1 and c_2 are fixed).
- AXX_b is right-angled too and $AX > c_1$.
- XBX_a is right-angled too and $XB > c_2$.
- This causes the inequality!
- $a_1^2 + b_1^2 + a_2^2 + b_2^2 \geq c_1^2 + c_2^2$
  The sum of the areas of the squares on a and b is greater than or equal to the sum of the squares on c.



Bithagoras DMLE I

- This was my *personal way of problem solving* as a teacher.
- Now we have to create a DMLE for *guided discovery* by the students, a didactic reconstruction.

A right triangle ABC is given with $\gamma = 90°$, along with a point X inside the triangle.
The perpendicular feet from X divide the sides a, b, c into a_1, a_2, b_1, b_2, and c_1, c_2.
Squares are constructed on these segments.
The areas of the squares on the legs are added together (blue), as are the areas of the squares on the hypotenuse (violet).
Drag X and compare the sums of the areas. Can you find a general statement?
In which case does equality hold?
For which position of X do we obtain the well-known Pythagorean figure?

- GeoGebra-Element Text.

Bithagoras DMLE II

A right triangle ABC is given with $\gamma = 90°$, along with a point X inside the triangle.
The feet of the perpendiculars from X divide the sides a, b, c into a_1, a_2, and b_1, b_2, and c_1, c_2.
Squares are constructed on these segments.
First, let X lie on side c = AB. Vary X in such a way that c_1 and c_2 remain unchanged
(e.g., using the up arrow key).
Display the auxiliary triangles and use them to justify that $a_1^2 + a_2^2 + b_1^2 + b_2^2 \geq c_1^2 + c_2^2$,
and that equality holds precisely when X lies on c = AB.

Aa π

Mögliche Antwort:
When X lies on c = AB, then c_2 = AX and $a_1^2 + b_1^2 = c_2^2$.
If X is moved perpendicular to c, then AX will become greater than c_2.
Then $AX^2 > c_2^2$.

The same applies to BX and c_1.

- GeoGebra-Element Question, with solution.

Conclusion

- **Dynamic Visualization** with GeoGebra files can help you investigate the problem and *find the solution*. It is more helpful than a static illustration. Problem solving process (first by the teacher).
- **Systematic variation** can answer the question WHY and can help you to *develop a proof*.
- **DMLE** can help your students to *learn and understand mathematics*. It enables guided problem solving for students.

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Epilog

“Thanks to computer graphics, much of the mathematician's search for patterns is now guided by what one can really see with the eye, whereas nineteenth-century mathematical giants like Gauss and Poincaré had to depend more on seeing with their mind's eye.”

Lynn A. Steen (1988). The Science of Patterns

Contact

GeoGebra Book: https://www.geogebra.org/**m/m3cxvryy**

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Thank You



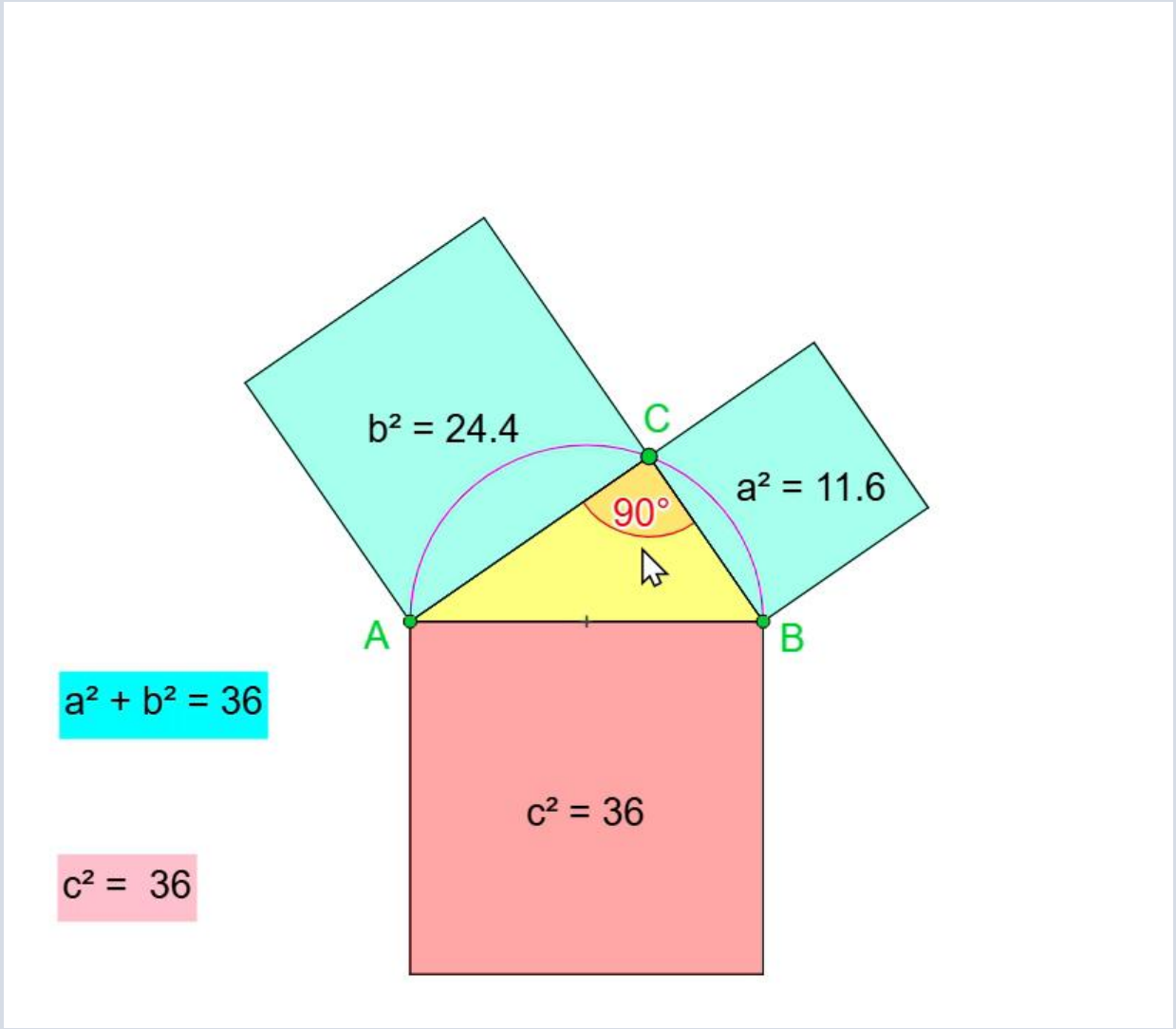
Appendix

Further Examples of dynamic mathematics and DMLE:

2. Dynamic Euclidean geometry: Pythagoras
3. Dynamic algebra & functions: Quadratic functions & equations
4. Dynamic calculus: Function Loupe & Differentiograph
5. Dynamic 3D geometry: Archimedean solids

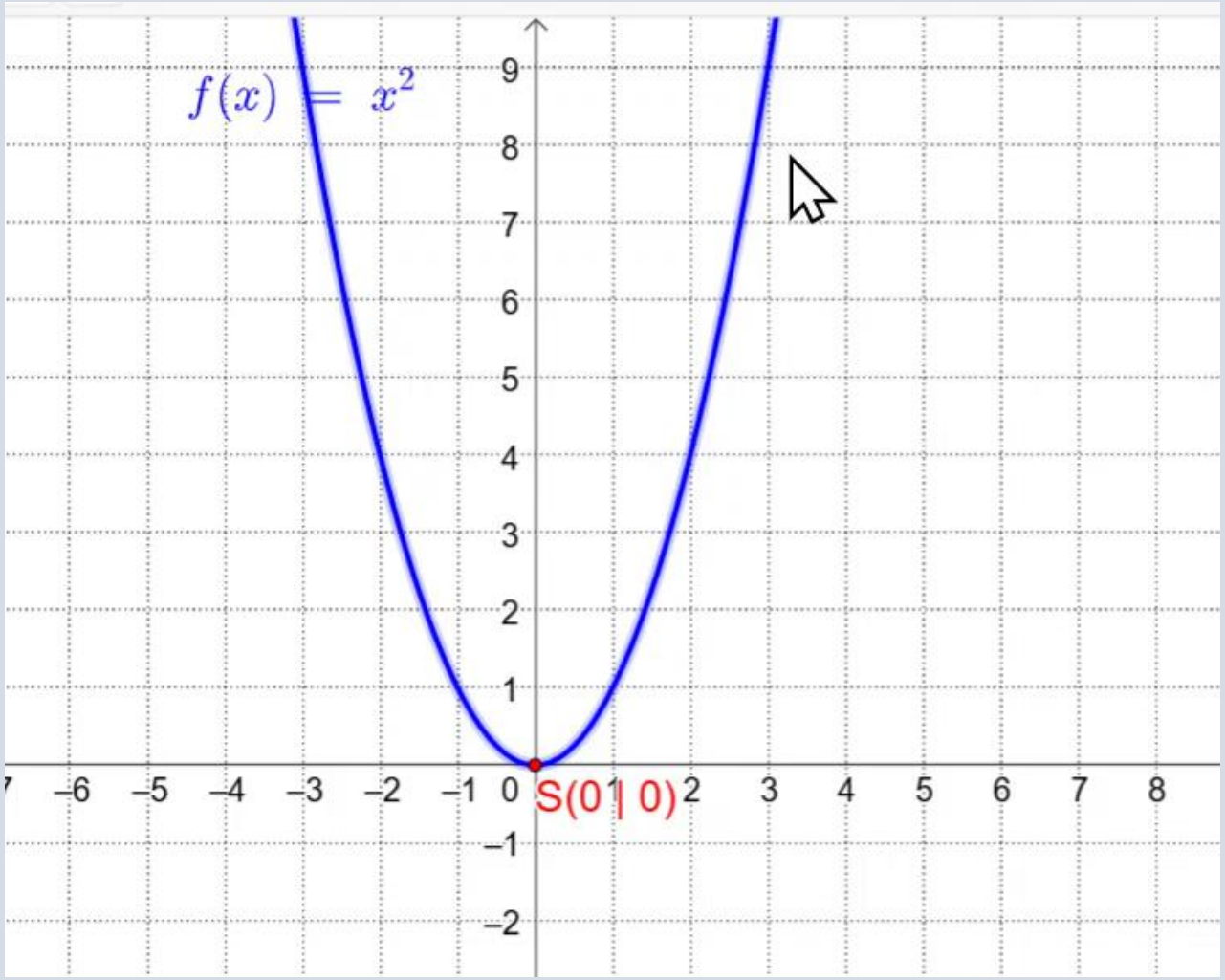
Example 2: The Pythagorean Theorem

- Formulated for right-angled triangles.
  One special case.
- Dynamization: 3 cases, qualitative version of the Cosine Theorem.



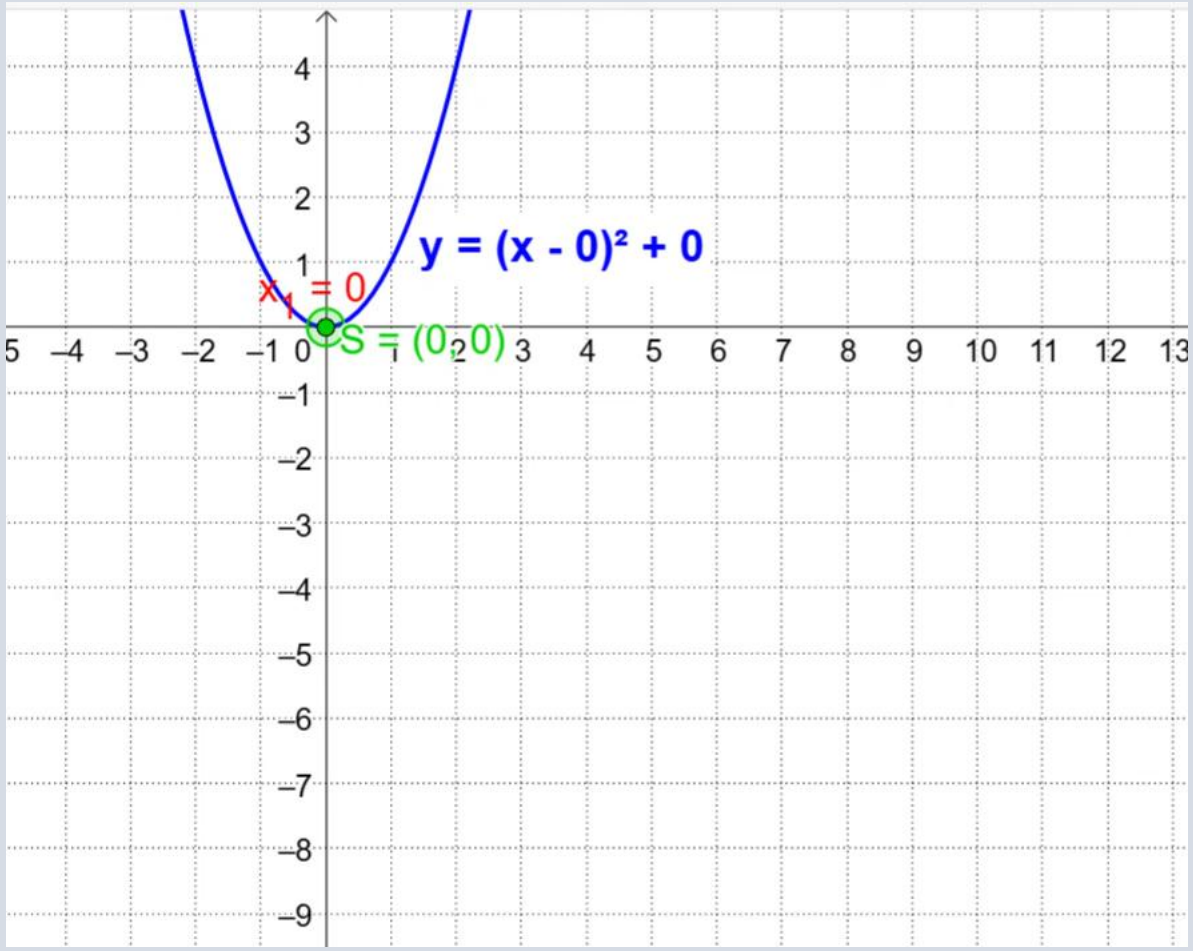
Example 3a: Quadratic Functions

- Detecting the vertex form $f(x) = (x - x_S)^2 + y_S$.



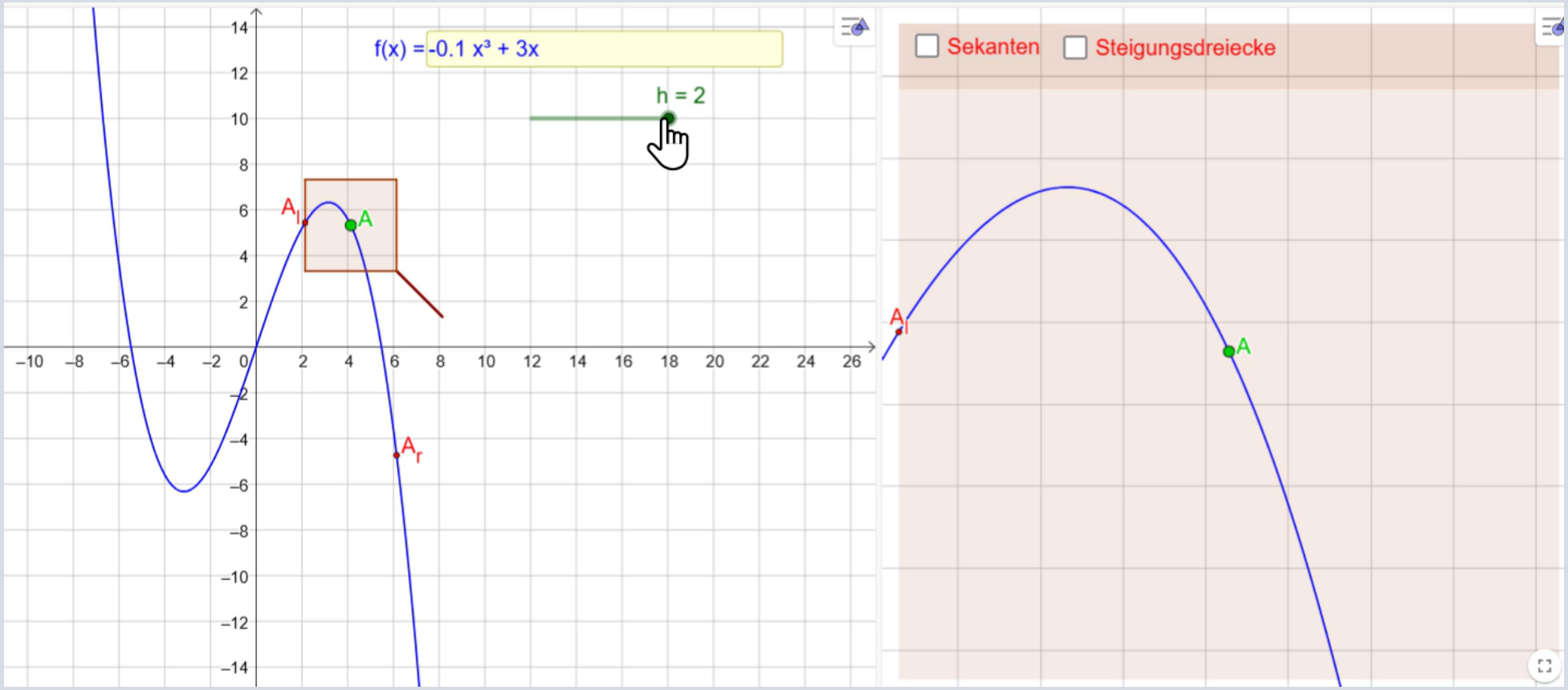
Example 3b: Quadratic Equations

- A formula for the solution of $(x - x_S)^2 + y_S = 0$-



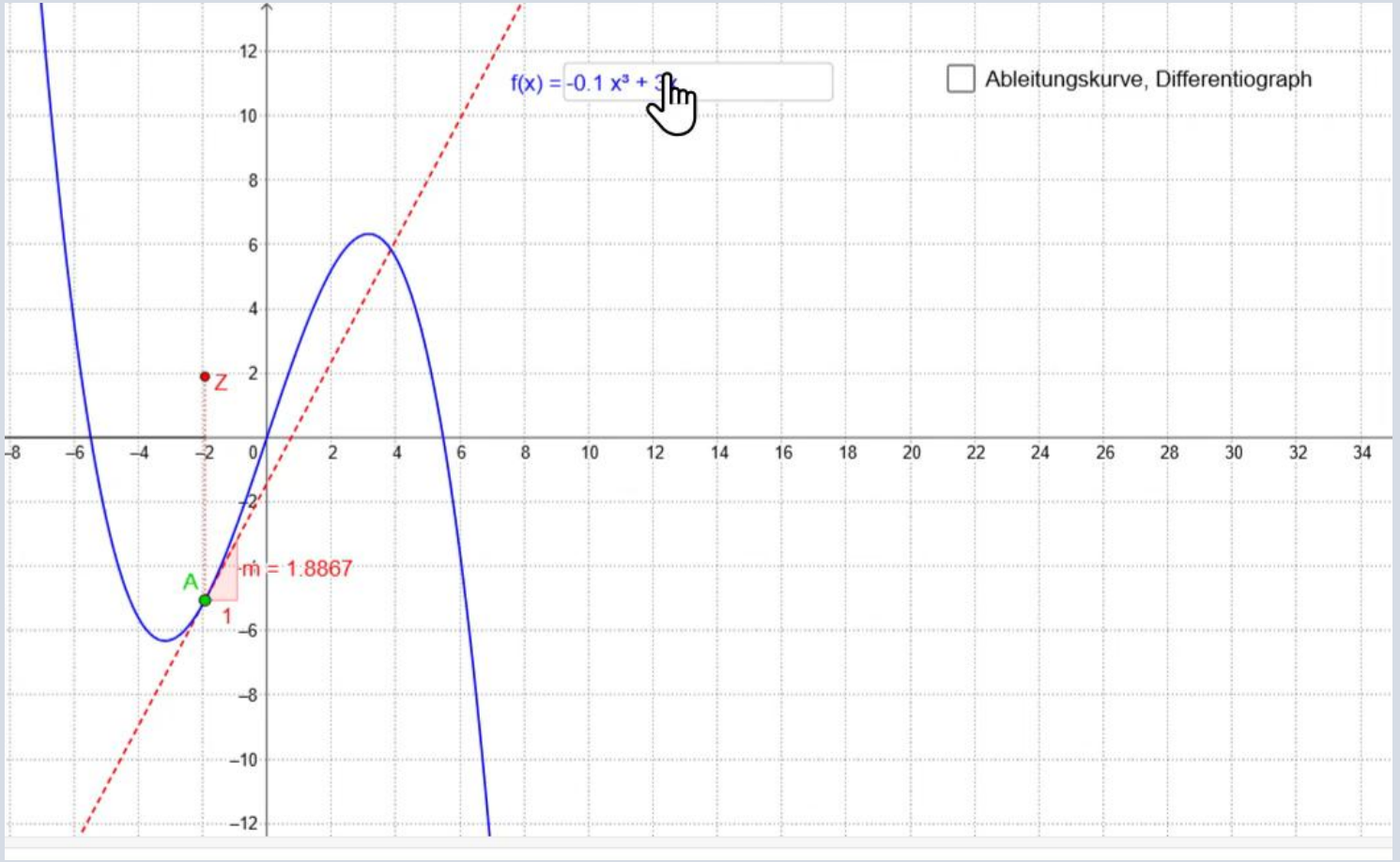
Example 4a: Visual Calculus (Function Loupe)

- Understanding the slope by zooming with the Function Loupe. Fundamental Idea of Local Linearity.



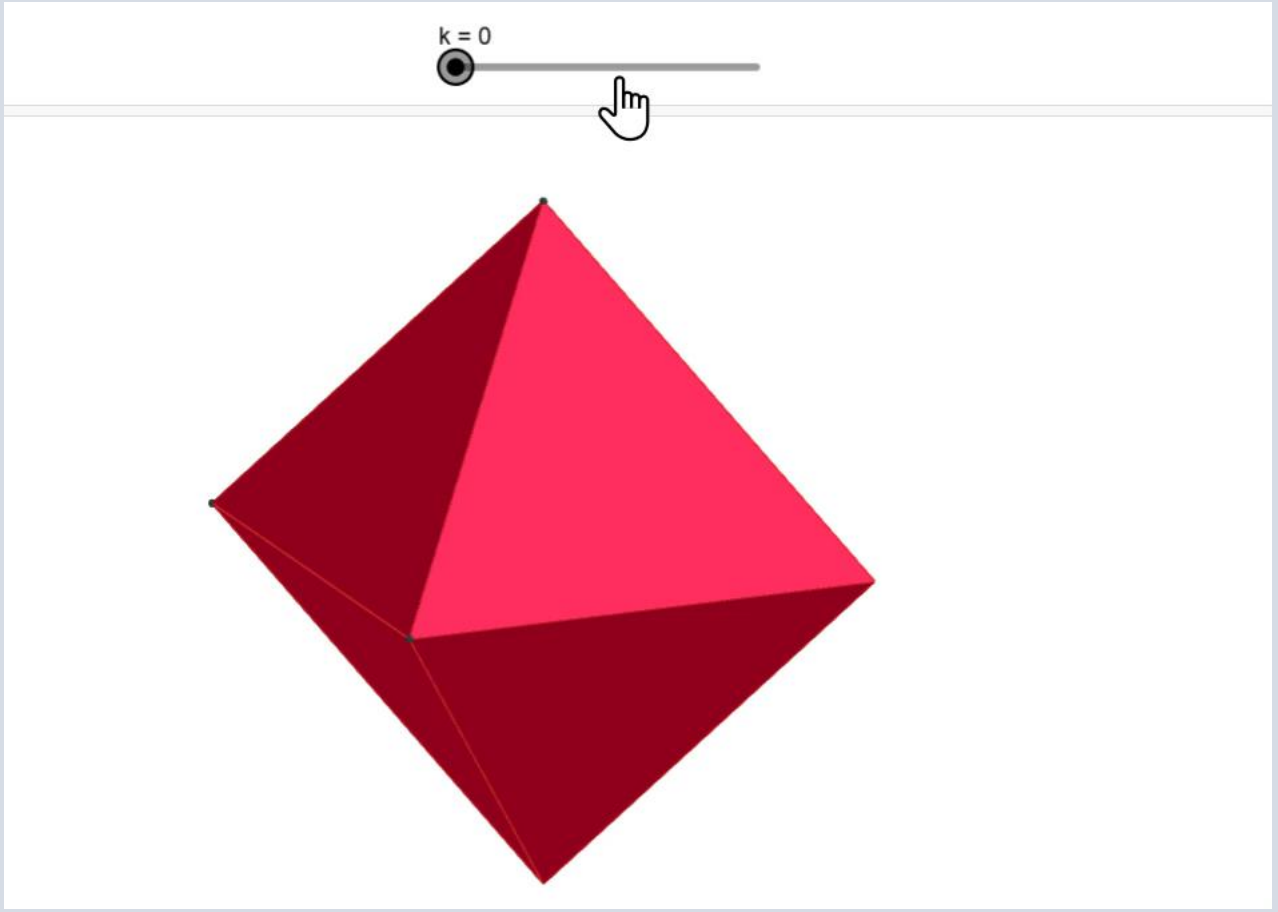
Example 4b: Visual Calculus (Differentiograph)

- The Differentiograph was a mechanical instrument for drawing the slope curve of a given function without Calculus. This is a digital simulation.



Example 5: Archimedean Solids, Truncating

- If we truncate Platonic solids, we get convex solids, which are in special cases Archimedean solids.



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