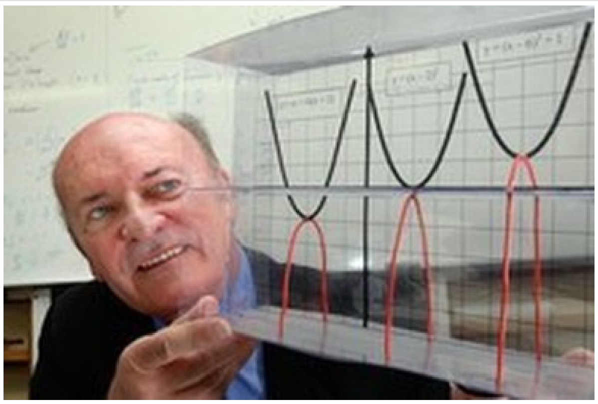
PHANTOM GRAPHS (/)



Mathematics teacher Philip Lloyd with a model of his "Phantom Parabolas" showing the real position of imaginary solutions of equations.

Very few people wake excitedly every Sunday at 3am thinking about calculus! But that is what happened to Epsom Girls Grammar teacher Philip Lloyd, who has come up with a new way of showing the real positions of imaginary solutions of equations.

The teacher of 47 years is now receiving international praise for his concept.

"It came to me at 3am on a Sunday morning," Mr Lloyd said.

"In simple terms, the solutions of an equation are where its graph crosses the x axis. Some graphs do not cross the x axis but we still say they have solutions which people call 'imaginary'."

It was this "imaginary" concept which many students struggle to accept. Because they can't see it, many tended to find it difficult to believe.

"I found that graphs have 'extra bits' on them which I call **'phantom graphs'** and these actually do cross the x axis.
In fact, I found that the **imaginary** solutions are at **real** places."

"I'd think of one type of graph one week and the next week something else would pop up. This continued for weeks. It was a very exciting time!"

"I would get up in the morning and I'd start making these models."

After spending most of his school holidays on the perspex models, Mr Lloyd demonstrated the new concept to his students.

Suddenly they could see what he was talking about and they "absolutely loved it".

Mr Lloyd has been using his models for several years now with great success.

Not only do the students love it, but his concept is gaining momentum in mathematical circles, so much so he was invited to share his idea at an international mathematics conference in South Africa in 2011 where a prominent mathematician from Cambridge University described Phantom Graphs as the "Highlight of the conference".
Philip has already given several presentations at universities in New Zealand.

A letter from the head of the conference says Mr Lloyd's paper on the concept is ***"Quite exceptional and exciting! It is a rare thing to see such a new idea in maths education!"***

In April 2012, Philip was the KEYNOTE SPEAKER at a Mathematics Symposium for "Innovations in Mathematics Education" held at GALWAY University, Ireland.

Click below to see an entertaining TV New Zealand

interview on Phantom Graphs.

http://www.youtube.com/watch?v=ctZ6gICQ4Pg
(http://www.youtube.com/watch?v=ctZ6gICQ4Pg)

Philip is available to give presentations for universities/colleges/teachers and students anywhere in the world.

You may contact Philip at philiplloyd1@gmail.com

The following 6 pages are a brief summary of PHANTOM GRAPHS produced in 2015 for people who would like an overview of the idea without too much complex mathematical theory.
Also you may wish to see this short 5 minute video as an introduction:
https://www.screencast.com/t/7Zum9tiW

(https://www.screencast.com/t/7Zum9tiW)

INTRODUCTION TO "PHANTOM GRAPHS".

This is the basic graph of $y = x^2$ and if we only use **real values** of x we only obtain **positive** values of y. **Fig 1**

$x = \pm 1$ we get $y = 1$
$x = \pm 2$ we get $y = 4$
$x = \pm 3$ we get $y = 9$

However, if we allow values of x such as:

$x = \pm i$ we get $y = -1$
$x = \pm 2i$ we get $y = -4$
$x = \pm 3i$ we get $y = -9$

Fig 1

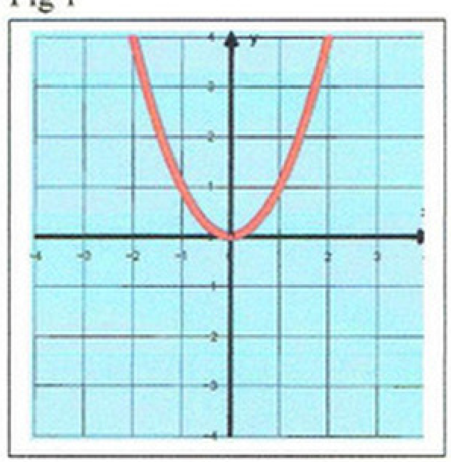
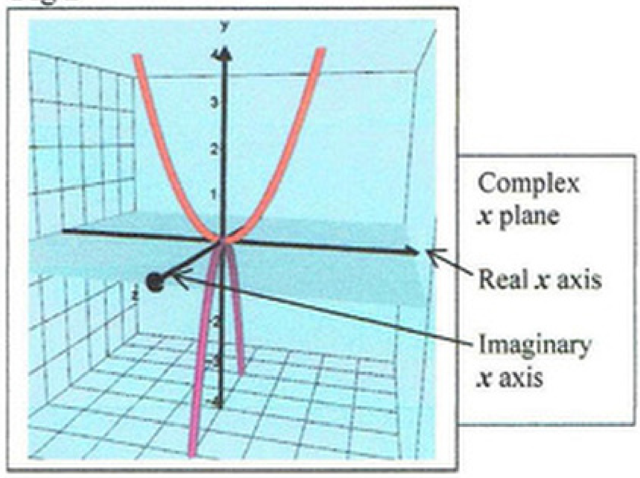


Fig 2



The **insight**, is to allow a **complex *x* PLANE** but with just a **real *y* AXIS. Fig 2**

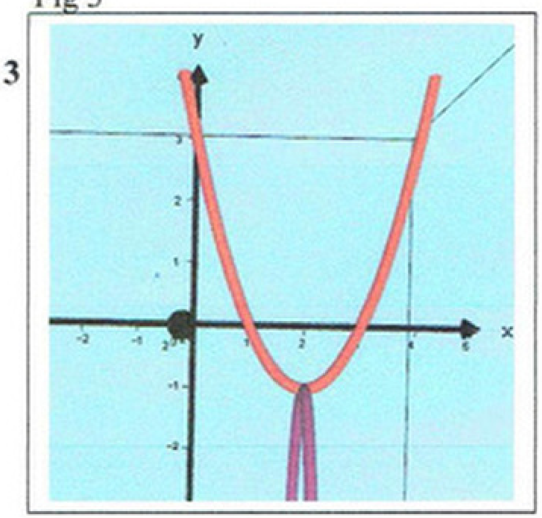
This produces a sort of "*phantom*" parabola underneath the basic parabola and at right angles to it.

I have discovered that nearly ALL curves have these extra "*phantom*" parts and more importantly, this has an intriguing connection with the **Fundamental Theorem of Algebra**.

Basically, the **Fundamental Theorem of Algebra** states that polynomial equations of the form: $ax^n + bx^{n-1} + cx^{n-2} + \dots\ px^2 + qx + r = 0$ will have n solutions.
(where n is a positive integer)

This is often interpreted as:
"*The solutions of an equation $f(x) = 0$ are where the graph of $y = f(x)$ crosses the **x** axis*"
but this only finds the solutions which are REAL numbers.

Fig 3



Consider the equation $x^2 - 4x + 3 = 0$
The graph of $y = x^2 - 4x + 3$ is as shown in **Fig 3**

The graph crosses the x axis at $x = 1$ and $x = 3$ so the solutions are $x = 1$ and 3

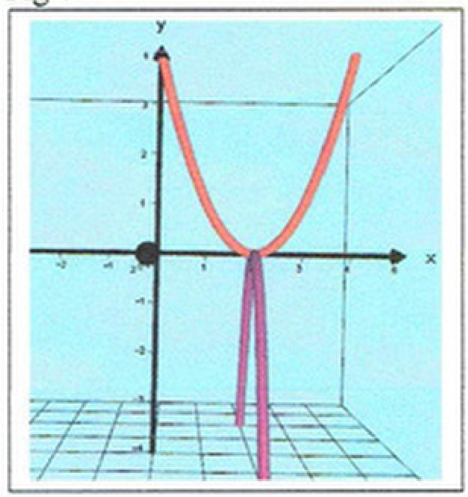
In this case, the phantom hanging below had no part to play in this logic.

However, consider the equation $x^2 - 4x + 4 = 0$
The graph of $y = x^2 - 4x + 4$ is as shown in Fig 4.

In this case, the top half of the parabola crosses the x axis at $x = 2$ **AND** the bottom half of the parabola (*the phantom*) also crosses the x axis at $x = 2$. ***(a double solution)***

The graph goes through the point (2, 0) twice.

Fig 4

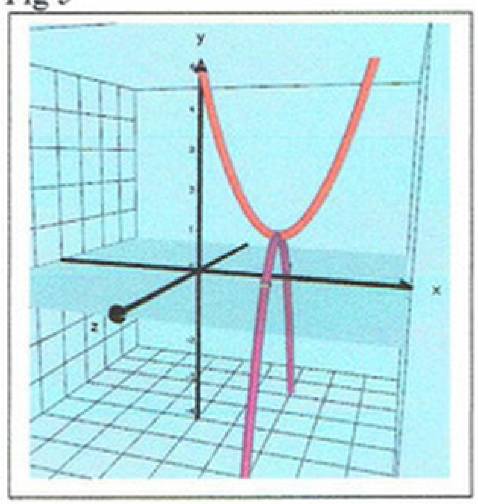


Of course, the most interesting case is when the basic **top half** of the parabola would not normally cross the ***x AXIS*** at all but its *phantom* **would** cross the ***complex x PLANE***!

Consider the equation $x^2 - 4x + 5 = 0$
The graph of $y = x^2 - 4x + 5$ is as shown in Fig 5.

The *phantom* crosses the ***x plane*** at $x = 2 + i$ *and* $x = 2 - i$ as shown in Fig 5 and these are the complex solutions of the equation.

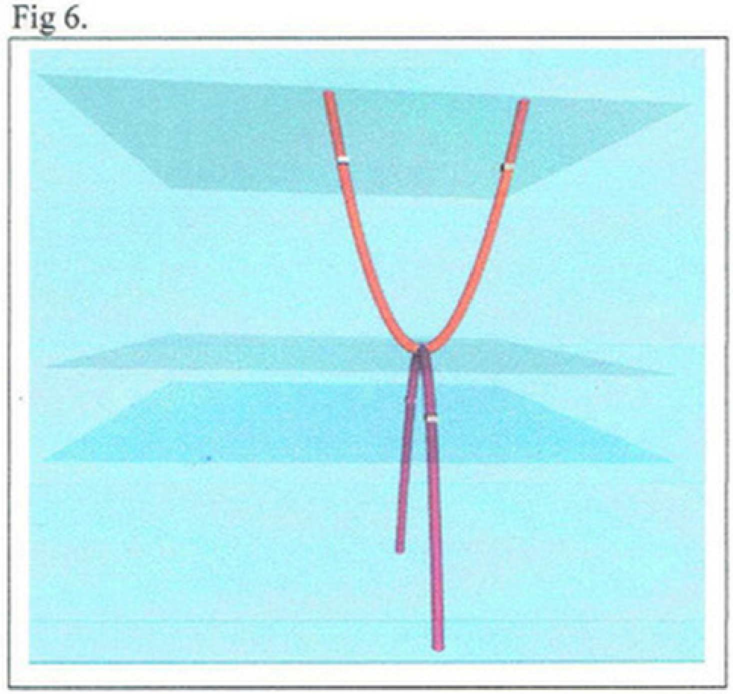
Fig 5



We can now re-state the **Fundamental Theorem of Algebra** as:
"The solutions of an equation ***f(x) = 0, whether they are real or complex,*** are where the graph of $y = f(x)$ crosses the ***complex x plane***".

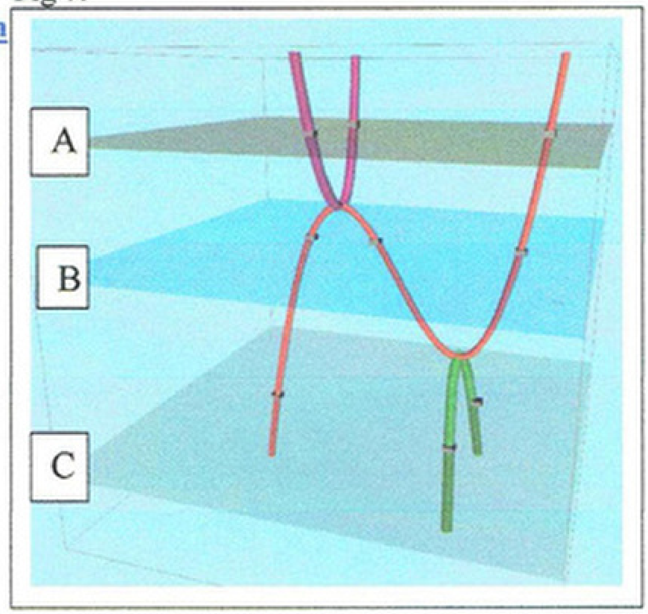
Clearly, we can see that any parabola (with its phantom) of the form:
$y = ax^2 + bx + c$
will cross **any** horizontal plane (which represents any real y value) **exactly two times. Fig 6.**

Fig 6.



In my website www.phantomgraphs.weebly.com you will find detailed working to show how cubic functions of the form:
$y = ax^3 + bx^2 + cx + d$
each have 2 phantoms emanating from their **maximum** and **minimum** points. See **Fig 7**.

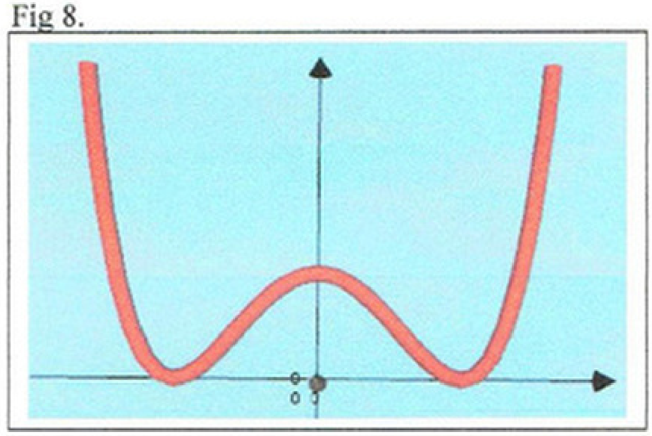
Fig 7.



We know that any Cubic equation of the form:
$ax^3 + bx^2 + cx + d = 0$ will have 3 solutions.

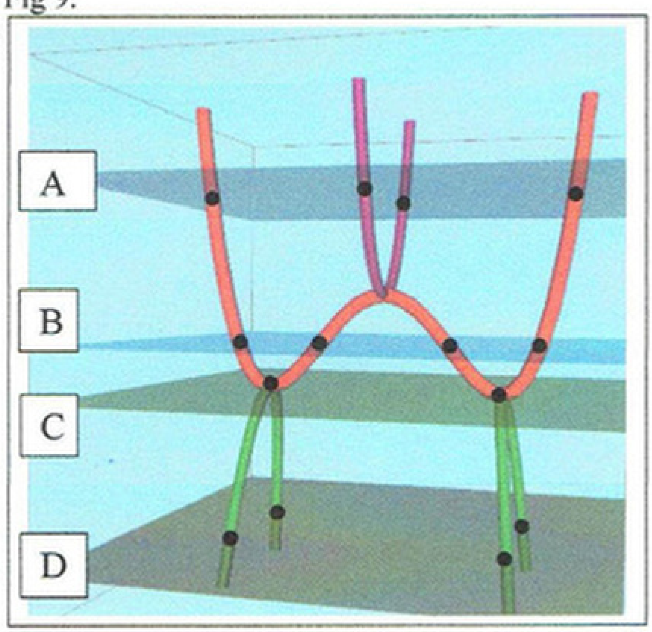
Sometimes we have **3 REAL solutions** as for the intersections with the middle **Plane B** in **Fig 7** and sometimes we have **1 real solution and 2 complex solutions** as on **Planes A and C.** See **Fig 7**.

Fig 8.



This is a typical Quartic graph. **Fig 8** showing 1 maximum and 2 minimum points.

Fig 9.



This is the same Quartic graph with its 3 *phantoms* emanating from each turning point. **Fig 9**.
The intersection points with the 4 planes A, B, C and D with the graph, are marked.
Plane A shows 2 real solutions on the basic RED curve and 2 imaginary solutions on the PURPLE *phantom.*

Plane B shows 4 real solutions on the basic RED curve.

Plane C shows 2 double real solutions which lie on the basic red curve and the GREEN *phantoms*.

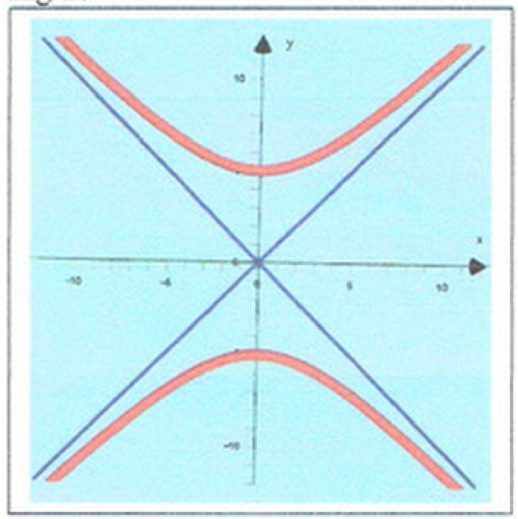
Plane D shows 4 imaginary solutions on the two GREEN *phantoms*. (2 sets of conjugate roots)

Clearly a Quartic curve will pass through ANY horizontal plane 4 times.

In my website www.phantomgraphs.weebly.com I have found that all sorts of graphs, not just polynomials, have some amazing and surprising *phantoms*.
Examples are $y = \dfrac{2x}{x^2 - 1}$, $y = \cos(x)$, $y = e^x$, $y^2 = x(x-3)^2$ and many more.

One particularly lovely surprise was the hyperbola $y^2 = x^2 + 25$ **Fig 10**

Fig 10



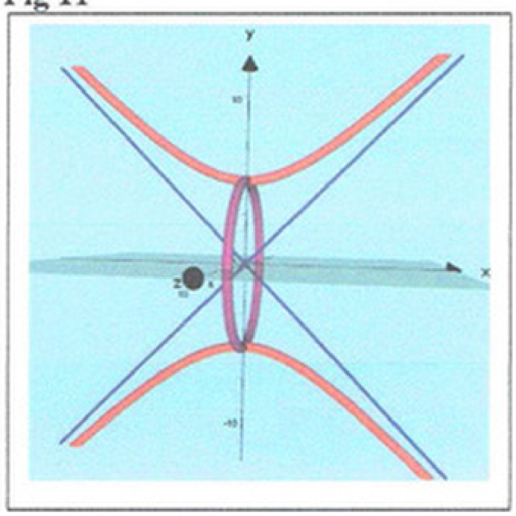
There are clearly NO real y values in the interval $-5 \leq y \leq 5$
So I decided to calculate complex x values for y values such as $y = 3$

substituting in $x^2 + 25 = y^2$
we obtain $x^2 + 25 = 9$
$x^2 = -16$
so $x = \pm 4i$

Similarly, if $y = 4$, $x = \pm 3i$
and if $y = 0$, $x = \pm 5i$

These of course are points on a CIRCLE of radius 5 units and this *phantom circle* joins the two halves of the hyperbola! See Fig 11.

Fig 11



When I first worked on ***phantom graphs*** I used to calculate the complex **points** as above and I made Perspex models to demonstrate the graphs clearly in 3 dimensions.
In order to draw the graphs in ***Autograph*** I had to work out the actual ***equations*** of the *phantoms*.

I will demonstrate the method for the hyperbola above.

Firstly we have to allow ***complex x*** values so that when x values appear in the equation we need to replace them with $(x + iz)$

The above equation becomes $y^2 = (x + iz)^2 + 25$ ***Equation 1***

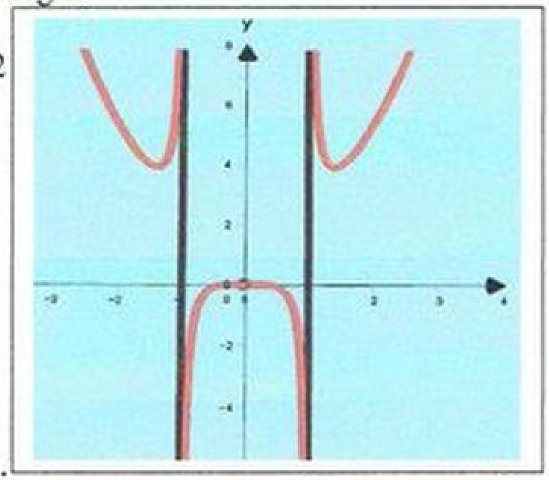
Expanding and rearranging: $y^2 = (x^2 - z^2 + 25) + (2xz)i$

The important idea now is that phantom graphs can have *complex x values* BUT the *y values* must only be REAL numbers.

This means the imaginary part of y must be zero.
That is $2xz = 0$ ***so*** $x = 0$ ***or*** $z = 0$
If $z = 0$ then ***Equation 1*** simply becomes $y^2 = x^2 + 25$ which is the original hyperbola.
If $x = 0$ then ***Equation 1 becomes*** $y^2 = (iz)^2 + 25$ ***that is*** $y^2 = -z^2 + 25$ ***or in its more familiar form*** $y^2 + z^2 = 25$ ***which is the phantom circle joining the two halves of the hyperbola!***

By far the most challenging complex algebra was needed in finding the equations of the *phantoms* for the function $y = \dfrac{x^4}{x^2 - 1}$ see Fig 12

Fig 12

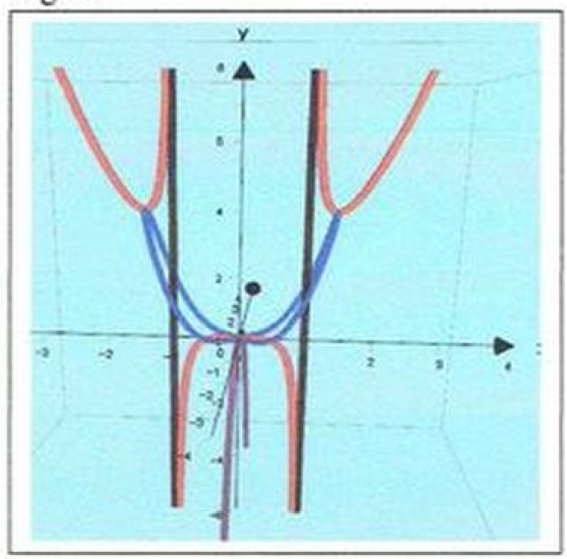


We can see that there are no real values of y in the interval $0 \le y \le 4$

Interestingly, if we consider a general y value such as $y = c$ we get $\dfrac{x^4}{x^2 - 1} = c$

which produces a typical quartic equation $x^4 - cx^2 + c = 0$ which of course has 4 solutions. If we draw $y = 5$ on **Fig 12** it crosses 4 times but if we draw $y = -2$ it only crosses twice.

Fig 13



But when we consider the graph with its Phantoms, we see that **any horizontal plane** $y = c$ crosses the graph 4 times which further verifies the truth of the Fundamental Theorem of Algebra. **Fig 13**

Incidentally, the equation of the bottom purple phantom is: $y = 1 - z^2 - \dfrac{1}{z^2 + 1}$

and the equation of the top blue phantom is:
$$y = x^2 - z^2 + 1 + \frac{(x^2 - z^2 - 1)}{(x^2 - z^2 - 1)^2 + 4x^2z^2}$$

The following section is the original presentation material I used at the International Mathematics Conference at Rhodes University in South Africa in 2011.
The photographs are of the perspex models of my "*phantom*" graphs. This was before I found the wonderful 3D graphing program called AUTOGRAPH.

PHANTOM GRAPHS.

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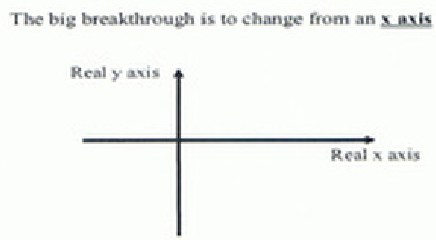
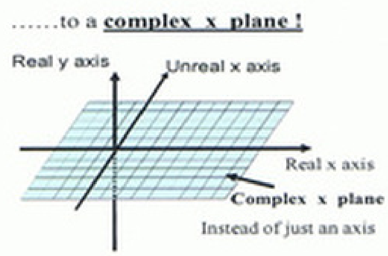
Abstract. While teaching "solutions of quadratics" and emphasising the idea that, in general, the solutions of $ax^2 + bx + c = 0$ are obviously where the graph of $y = ax^2 + bx + c$ crosses the x axis, I started to be troubled by the special case of parabolas that do not even cross the x axis. We say these equations have "complex solutions" but **physically, where are these solutions**?
With a little bit of lateral thinking, I realised that **we can physically find the actual positions of the complex solutions of any polynomial equation** and indeed many other common functions! The theory also shows clearly and pictorially, why the complex solutions of polynomial equations with real coefficients occur in conjugate pairs.

Fig 1



Introduction. Consider the graph $y = x^2$.
We normally just find the positive y values such as: $(\pm1, 1)$, $(\pm2, 4)$, $(\pm3, 9)$ but we can also find **negative y values** even though the graph does not seem to exist under the x axis:
If $y = -1$ then $x^2 = -1$ and $x = \pm i$.
If $y = -4$ then $x^2 = -4$ and $x = \pm 2i$.
If $y = -9$ then $x^2 = -9$ and $x = \pm 3i$

Fig 2



Thinking very laterally, I thought that instead of just having a ***y axis*** and an ***x AXIS*** (as shown in **Fig 1**) we should have a ***y axis*** but a **complex x PLANE!** (as shown on **Fig 2**)
This means that the usual form of the parabola $y = x^2$ exists in the normal x, y plane but another part of the parabola exists at right angles to the usual graph. See **Fig 3**.
Fig 4 is a Perspex model of $y = x^2$ and its "phantom" hanging at right angles to it.

Fig 3

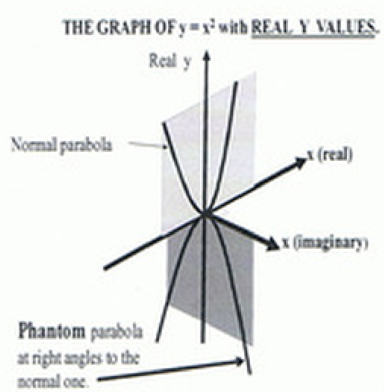
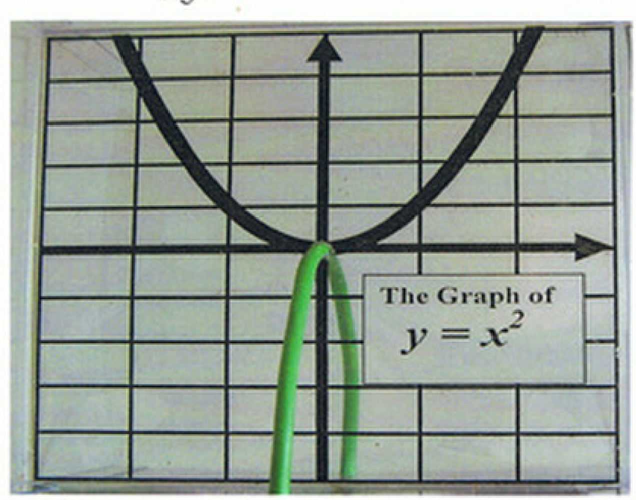


Fig 4



"PHANTOM GRAPHS". Now let us consider the graph $y = (x - 1)^2 + 1 = x^2 - 2x + 2$

The minimum real y value is normally thought to be 1
but now we can have any real y values!

If $y = 0$ then $(x - 1)^2 + 1 = 0$
so that $(x - 1)^2 = -1$
producing $x - 1 = \pm i$
therefore $x = 1 + i$ *and* $x = 1 - i$

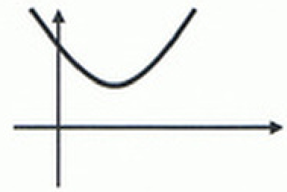
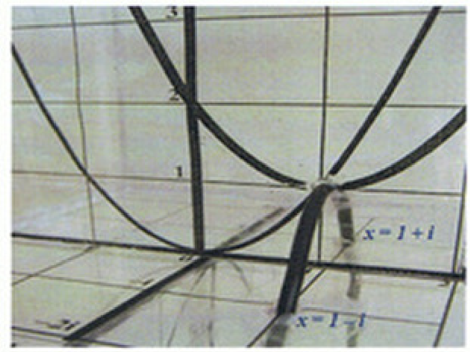


Fig 5

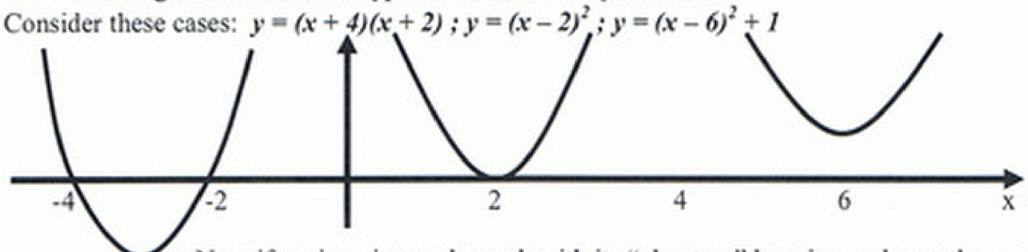


If $y = -3$ then $(x - 1)^2 + 1 = -3$
so that $(x - 1)^2 = -4$
therefore $x = 1 + 2i$ *and* $x = 1 - 2i$

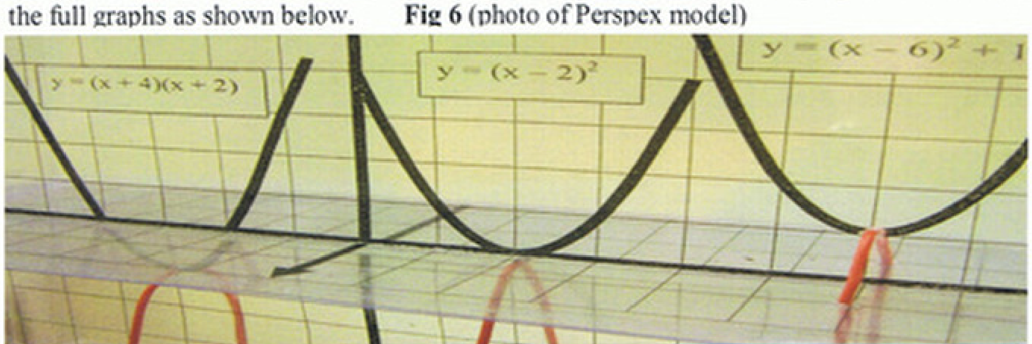
Similarly if $y = -8$ then $(x - 1)^2 + 1 = -8$
so that $(x - 1)^2 = -9$
therefore $x = 1 + 3i$ *and* $x = 1 - 3i$

The result is another "phantom" parabola which is "hanging" from the normal graph $y = x^2 - 2x + 2$ and the exciting and fascinating part is that the **solutions of** $x^2 - 2x + 2 = 0$ are $1 + i$ and $1 - i$ which are **where the graph crosses the x plane**! See **Fig 5**

In fact ALL parabolas have these "phantom" parts hanging from their lowest points and at right angles to the normal x, y plane.
It is interesting to consider the 3 types of solutions of quadratics.
Consider these cases: $y = (x + 4)(x + 2)$; $y = (x - 2)^2$; $y = (x - 6)^2 + 1$



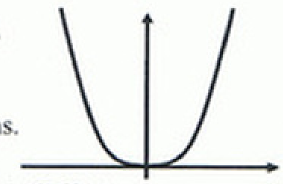
Now if we imagine each graph with its "phantom" hanging underneath we get the full graphs as shown below. **Fig 6** (photo of Perspex model)



Here the "phantom" has no effect on the solutions $x = -4$ *and* $x = -2$.	Notice that the curves go through the point $x = 2$ twice! (a double solution)	The solutions are where the graph crosses the x ***plane*** at $x = 6 \pm i$ **(a conjugate pair)**

Now consider the graph of $y = x^4$

We normally think of this as just a U shaped curve as shown. This consists of points (0, 0), (±1, 1), (±2, 16), (±3, 81) etc The fundamental theorem of algebra tells us that equations of the form $x^4 = c$ should have 4 solutions not just 2 solutions.

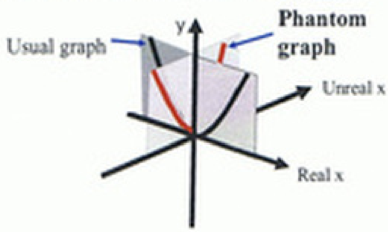


If $y = 1$, $x^4 = 1$ so using De Moivre's Theorem: $r^4 cis\ 4\theta = 1 cis\ (360n)$
$r = 1$ and $4\theta = 360n$ therefore $\theta = 0, 90, 180, 270$ producing the 4 solutions :
$x_1 = 1\ cis\ 0 = 1,\ x_2 = 1\ cis\ 90 = i,\ x_3 = 1\ cis\ 180 = -1$ and $x_4 = 1\ cis\ 270 = -i$

If $y = 16$, $x^4 = 16$ so using De Moivre's Theorem: $r^4 cis\ 4\theta = 16 cis\ (360n)$
$r = 2$ and $4\theta = 360n$ therefore $\theta = 0, 90, 180, 270$ producing the 4 solutions :
$x_1 = 2\ cis\ 0 = 2,\ x_2 = 2\ cis\ 90 = 2i,\ x_3 = 2\ cis\ 180 = -2,\ x_4 = 2\ cis\ 270 = -2i$

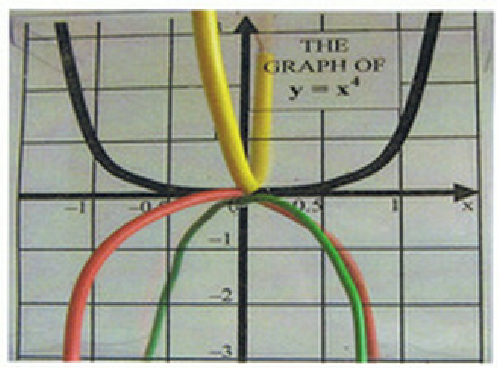
Fig 7

This means $y = x^4$ has another **phantom** part at right angles to the usual graph.



The points (1, 1), (–1, 1), (2, 16), (–2, 16) will produce the ordinary graph but the points (i, 1), (–i, 1), (2i, 16), (–2i, 16) will produce a similar curve at right angles to the ordinary graph. **Fig 7**

Fig 8 (photo of Perspex model)



But this is not all!

We now consider **negative** real y values!

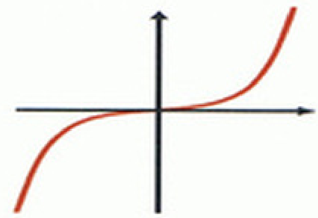
Consider $y = -1$ so $x^4 = -1$
Using De Moivre's Theorem:
$r^4 cis\ 4\theta = 1 cis\ (180 + 360n)$
$r = 1$ and $4\theta = 180 + 360n$ so $\theta = 45 + 90n$
$x_1 = 1\ cis\ 45,\quad x_2 = 1\ cis\ 135,$
$x_3 = 1\ cis\ 225,\ x_4 = 1\ cis\ 315$

Similarly, if $y = -16$, $x^4 = -16$
Using De Moivre's Theorem:
$r^4 cis\ 4\theta = 16 cis\ (180 + 360n)$
$r = 2$ and $4\theta = 180 + 360n$ so $\theta = 45 + 90n$
$x_1 = 2\ cis\ 45,\ x_2 = 2\ cis\ 135,$
$x_3 = 2\ cis\ 225,\ x_4 = 2\ cis\ 315$

The points corresponding to negative y values produce two curves identical in shape to the two curves for positive y values but they are rotated 45 degrees as shown on **Fig 8.**

NOTE: Any horizontal plane crosses the curve in 4 places because all equations of the form $x^4 = \pm c$ have **4** solutions and it is clear from the photo that the solutions are conjugate pairs!

Consider the basic cubic curve $y = x^3$.



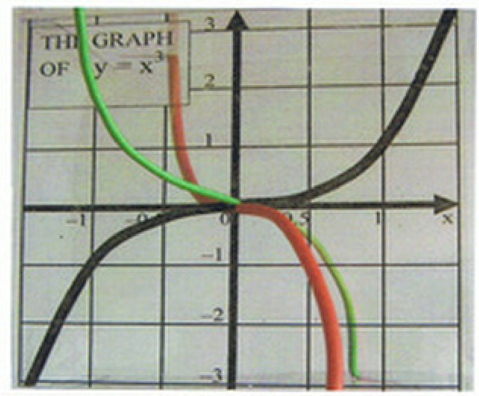
Equations with x^3 have 3 solutions.
If $y = 1$ then $x^3 = 1$
so $r^3 cis\, 3\theta = 1 cis\,(360n)$
$r = 1$ and $\theta = 120n = 0, 120, 240$
$x_1 = 1\ cis\ 0,\ x_2 = 1\ cis\ 120,\ x_3 = 1\ cis\ 240$

Similarly if $y = 8$ then $x^3 = 8$
so $r^3 cis\, 3\theta = 8 cis\,(360n)$
$r = 2$ and $\theta = 120n = 0, 120, 240$
$x_1 = 2\ cis\ 0,\ x_2 = 2\ cis\ 120,\ x_3 = 2\ cis\ 240$

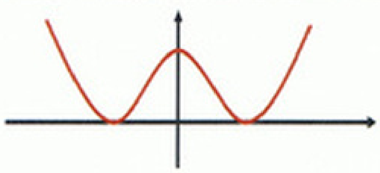
Also y can be negative. If $y = -1$, $x^3 = -1$
so $r^3 cis\, 3\theta = 1 cis\,(180 + 360n)$
$r = 1$ and $3\theta = 180 + 360n$ so $\theta = 60 + 120n$
$x_1 = 1\ cis\ 60,\ x_2 = 1\ cis\ 180,\ x_3 = 1\ cis\ 300$

The result is THREE identical curves situated at 120 degrees to each other! (See Fig 9)

Fig 9 (photo of Perspex model)



Now consider the graph $y = (x+1)^2(x-1)^2 = (x^2-1)(x^2-1) = x^4 - 2x^2 + 1$



Any horizontal line (or plane) should cross this graph at 4 places because any equation of the form $x^4 - 2x^2 + 1 = C$ (where C is a constant) has 4 solutions.

If $x = \pm 2$ then $y = 9$
so solving $x^4 - 2x^2 + 1 = 9$
we get : $x^4 - 2x^2 - 8 = 0$
so $(x+2)(x-2)(x^2+2) = 0$
giving $x = \pm 2$ and $\pm\sqrt{2}\, i$

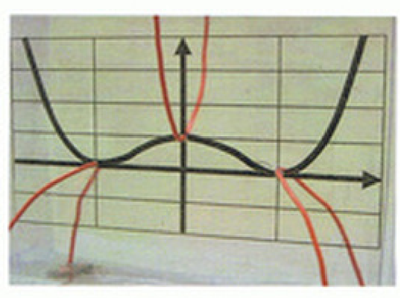
Similarly if $x = \pm 3$ then $y = 64$
so solving $x^4 - 2x^2 + 1 = 64$
we get $x^4 - 2x^2 - 63 = 0$
so $(x+3)(x-3)(x^2+7) = 0$
giving $x = \pm 3$ and $\pm\sqrt{7}\, i$

The complex solutions are all of the form $\mathbf{0} \pm n\boldsymbol{i}$. This means that a **phantom curve**, **at right angles** to the basic curve, stretches upwards from the maximum point.

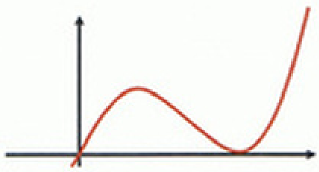
If $y = -1$, $x = -1.1 \pm 0.46i$, $1.1 \pm 0.46i$
If $y = -2$, $x = -1.2 \pm 0.6i$, $1.2 \pm 0.6i$
If $y = -4$, $x = -1.3 \pm 0.78i$, $1.3 \pm 0.78i$

Notice that the real parts of the x values vary. This means that the **phantom** curves hanging off from the two minimum points are not in a vertical plane as they were for the parabola. See **Fig 10. Clearly all complex solutions to $x^4 - 2x^2 + 1 = C$ are conjugate pairs.**

Fig 10 (photo of Perspex model)

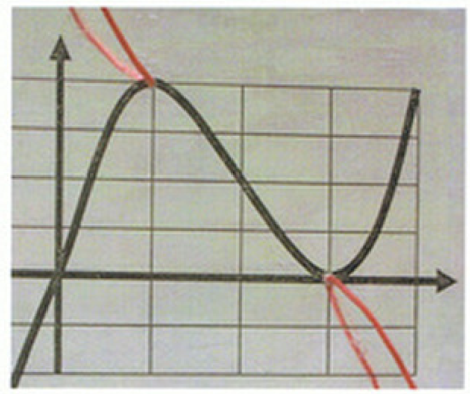


Consider the cubic curve $y = x\,(x - 3)^2$



As before, any horizontal line (or plane) should cross this graph at 3 places because any equation of the form :
$x^3 - 6x^2 + 9x =$ ***"a constant"***, has 3 solutions.

Fig 11 (photo of Perspex model)

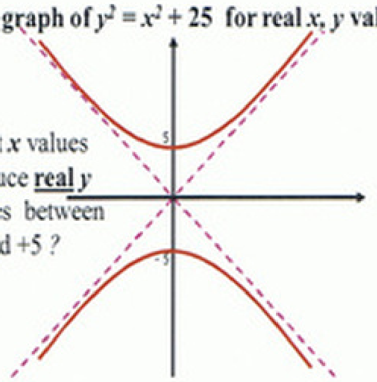


If $x^3 - 6x^2 + 9x = 5$ *then* $x = 4.1$ and $\underline{\mathbf{0.95} \pm 0.6i}$
If $x^3 - 6x^2 + 9x = 6$ *then* $x = 4.2$ and $\underline{\mathbf{0.90} \pm 0.8i}$
If $x^3 - 6x^2 + 9x = 7$ *then* $x = 4.3$ and $\underline{\mathbf{0.86} \pm 0.9i}$
So the left hand phantom is leaning to the left from the maximum point (1, 4).

If $x^3 - 6x^2 + 9x = -1$ *then* $x = -0.1$ and $\underline{\mathbf{3.05} \pm 0.6i}$
If $x^3 - 6x^2 + 9x = -2$ *then* $x = -0.2$ and $\underline{\mathbf{3.1} \pm 0.8i}$
If $x^3 - 6x^2 + 9x = -3$ *then* $x = -0.3$ and $\underline{\mathbf{3.14} \pm 0.9i}$
So the right hand phantom is leaning to the right from the minimum point (3, 0). See Fig 11

The HYPERBOLA $y^2 = x^2 + 25.$ This was the most surprising and absolutely delightful Phantom Graph that I found whilst researching this concept.

The graph of $y^2 = x^2 + 25$ for real x, y values.



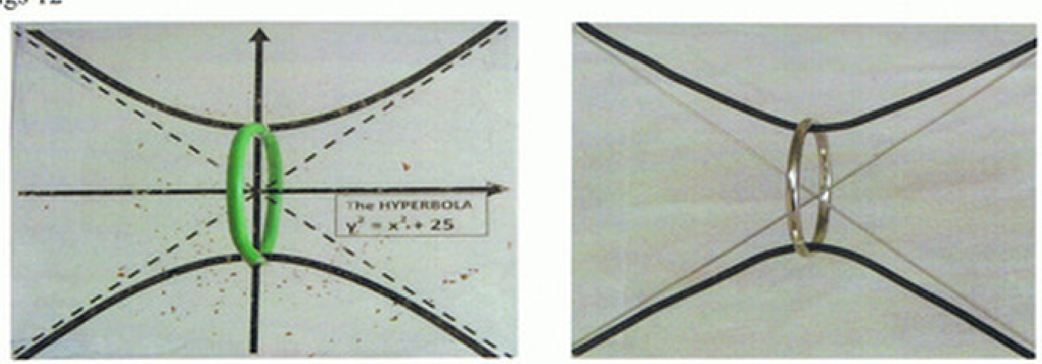
What x values produce **real** y values between -5 and +5 ?

If $y = 4$ then $16 = x^2 + 25$
and $-9 = x^2$
so $x = \pm 3i$
Similarly if $y = 3$ then $9 = x^2 + 25$
so $x = \pm 4i$
And if $y = 0$ then $0 = x^2 + 25$
so $x = \pm 5i$
These are points on a circle of radius 5 units.
(0, 5) $(\pm 3i, 4)$ $(\pm 4i, 3)$ $(\pm 5i, 0)$
The circle has complex x values but real y values.

This circle is in the plane at right angles to the hyperbola and joining its two halves!

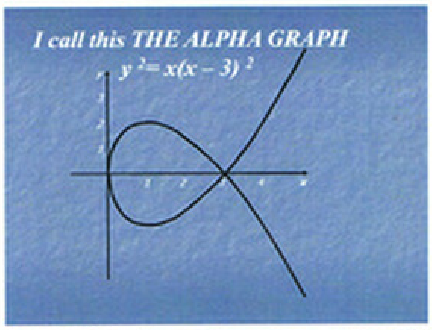
See Figs 12 for photos of the Perspex models.

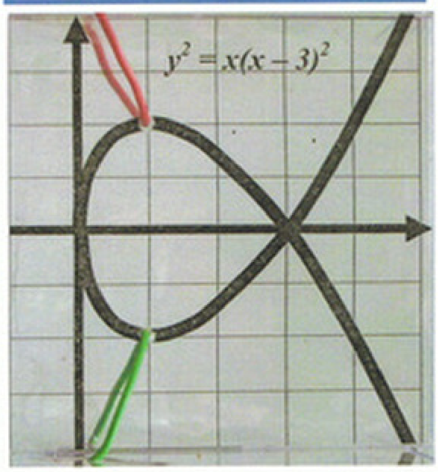
Figs 12



AFTERMATH!!!

I recently started to think about some other curves and thought it worthwhile to include them.





Using a technique from previous graphs:
I choose an x value such as $x = 5$,
calculate the y^2 value, ie $y^2 = 20$ and $y \approx 4.5$
then solve the equation $x(x-3)^2 = 20$
already knowing one factor is $(x-5)$

$$\begin{aligned} ie \qquad x(x-3)^2 &= 20 \\ x^3 - 6x^2 + 9x - 20 &= 0 \\ (x-5)(x^2 - x + 4) &= 0 \\ x = 5 \text{ or } \tfrac{1}{2} &\pm 1.9i \end{aligned}$$

This means (5, 4.5) is an "ordinary" point on the graph but two "phantom" points are ($\frac{1}{2} \pm 1.9i$, 4.5)

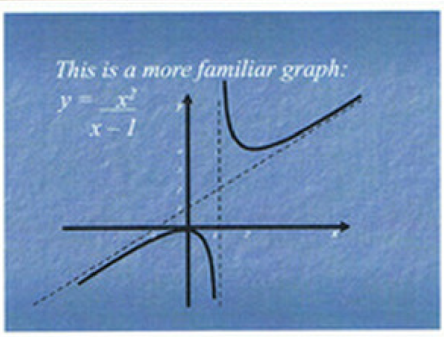
Similarly:
If $x = 6$, $y^2 = 54$ and $y = \pm 7.3$ *so*

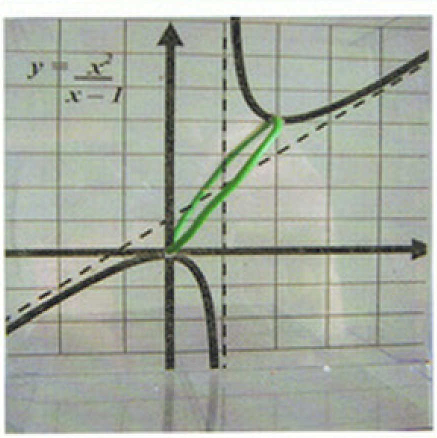
$$\begin{aligned} x(x-3)^2 &= 54 \\ x^3 - 6x^2 + 9x - 54 &= 0 \\ (x-6)(x^2 + 9) &= 0 \\ x = 6 \text{ or } &\pm 3i \end{aligned}$$

And
If $x = 7$, $y^2 = 112$ and $y = \pm 10.6$ *so*

$$\begin{aligned} x(x-3)^2 &= 112 \\ x^3 - 6x^2 + 9x - 112 &= 0 \\ (x-7)(x^2 + x + 16) &= 0 \\ x = 7 \text{ or } -\tfrac{1}{2} &\pm 4i \end{aligned}$$

Hence we get the two phantom graphs as shown.





$y = \dfrac{x^2}{x-1}$ **Here we need to find complex x values which produce real y values from 0 to 4.**

If $y = 0$ $\quad x = 0$

If $y = 1$ $\quad x = \dfrac{1}{2} \pm \dfrac{\sqrt{3}i}{2}$

If $y = 2$ $\quad x = 1 \pm i$

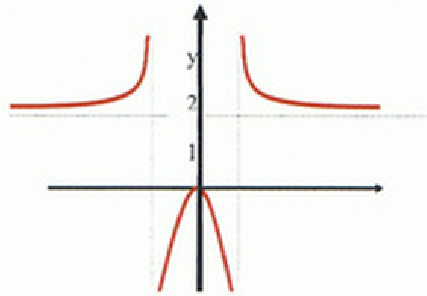
If $y = 3$ $\quad x = \dfrac{3}{2} \pm \dfrac{\sqrt{3}i}{2}$

If $y = 4$ $\quad x = 2$

These points produce the phantom "oval" shape as shown in the picture on the left.

Consider the graph $y = \dfrac{2x^2}{x^2 - 1} = 2 + \dfrac{2}{x^2 - 1}$

This has a horizontal asymptote $y = 2$
and two vertical asymptotes $y = \pm 1$



If y = 1 then $\dfrac{2x^2}{x^2 - 1} = 1$

so $2x^2 = x^2 - 1$

and $x^2 = -1$

producing $x = \pm i$

This implies there is a "phantom graph" which approaches the horizontal asymptotic plane y = 2 and is at right angles to the x, y plane, resembling an upside down normal distribution curve.

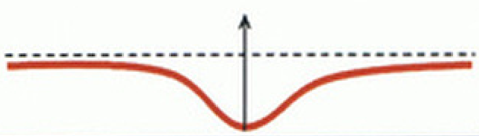
If y = 1.999 then $\dfrac{2x^2}{x^2 - 1} = 1.999$

so $2x^2 = 1.999x^2 - 1.999$

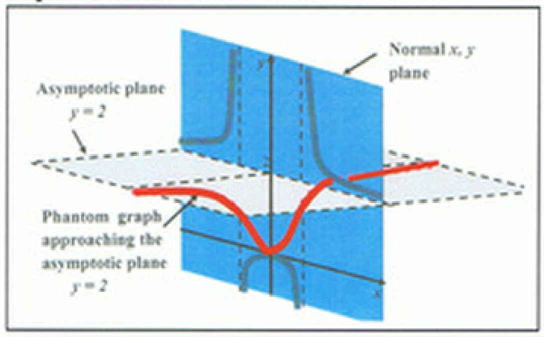
and $0.001x^2 = -1.999$

Producing $x^2 = -1999$

$x \approx \pm 45\, i$



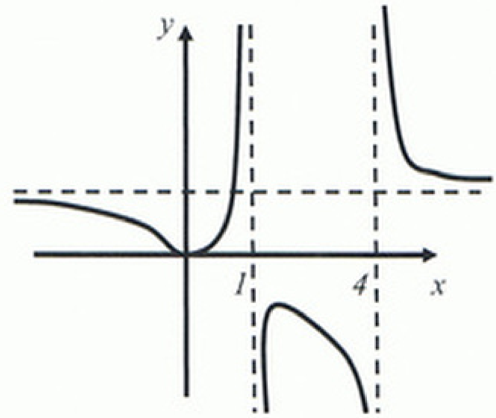
Side view of "phantom" approaching y = 2



Consider an apparently "similar" equation but with a completely different "Phantom".

$y = \dfrac{x^2}{(x-1)(x-4)} = \dfrac{x^2}{x^2 - 5x + 4}$

The minimum point is (0, 0)
The maximum point is (1.6, –1.8)



If $y = -0.1$, $x = 0.2 \pm 0.56i$

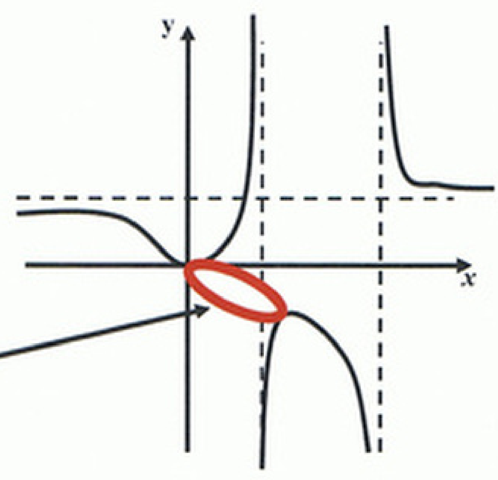
If $y = -0.2$, $x = 0.4 \pm 0.7i$

If $y = -0.5$, $x = 0.8 \pm 0.8i$

If $y = -1$, $x = 1.25 \pm 0.66i$

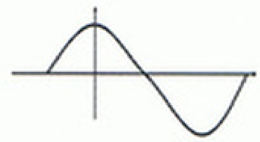
If $y = -1.5$, $x = 1.5 \pm 0.4i$

If $y = -1.7$, $x = 1.6 \pm 0.2i$



These results imply that a "phantom" ***oval*** *shape joins the minimum point (0, 0) to the maximum point (1.6, –1.78).*

The final two graphs I have included in this paper involve some theory too advanced for secondary students but I found them absolutely fascinating!



If $y = \cos(x)$ what about y values > 1 and < -1 ?

Using $\cos(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} - \ldots\ldots$

Let's find $\cos(\pm\, i) = 1 + \dfrac{1}{2!} + \dfrac{1}{4!} + \dfrac{1}{6!} + \dfrac{1}{8!} \ldots\ldots$

$$\approx \mathbf{1.54} \ \ \textbf{(ie > 1)}$$

Similarly $\cos(\pm\, 2i) = 1 + \dfrac{4}{2!} + \dfrac{16}{4!} + \dfrac{64}{6!} + \ldots$

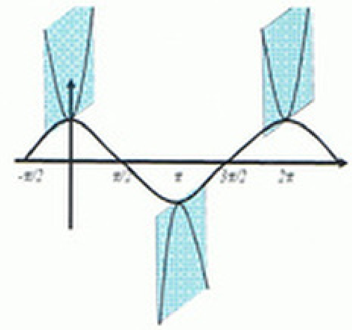
$$\approx \mathbf{3.8}$$

Also find $\cos(\pi + i) = \cos(\pi)\cos(i) - \sin(\pi)\sin(i)$

$$= -1 \times \cos(i) - 0$$

$$\approx \mathbf{-1.54} \ \ \textbf{(ie < -1)}$$

These results imply that the cosine graph also has its own "**phantoms**" in vertical planes at right angles to the usual x, y graph, emanating from each max/min point.



Finally consider the exponential function $y = e^x$. How can we find x if $e^x = -1$?

Using the expansion for $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \ldots\ldots$

We can find $e^{xi} = 1 + xi + \dfrac{(xi)^2}{2!} + \dfrac{(xi)^3}{3!} + \dfrac{(xi)^4}{4!} + \dfrac{(xi)^5}{5!} + \ldots\ldots$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$

$$= \quad (\cos x) \quad + \quad i\ (\sin x)$$

If we are to get **REAL y values** then using $e^{xi} = \cos x + i \sin x$, we see that $\sin x$ must be zero.

This only occurs when $x = 0, \pi, 2\pi, 3\pi, \ldots$ (or generally $n\pi$)

$e^{\pi i} = \cos\pi + i\sin\pi = {}^{-}1 + 0i$, $\quad e^{2\pi i} = \cos 2\pi + i\sin 2\pi = +1 + 0i$

$e^{3\pi i} = \cos 3\pi + i\sin 3\pi = {}^{-}1 + 0i$, $\quad e^{4\pi i} = \cos 4\pi + i\sin 4\pi = +1 + 0i$

Now consider $y = e^X$ where $X = x + 2n\pi i$ (ie even numbers of π)

ie $y = e^{x + 2n\pi i} = e^x \times e^{2n\pi i} = e^x \times 1 = e^x$

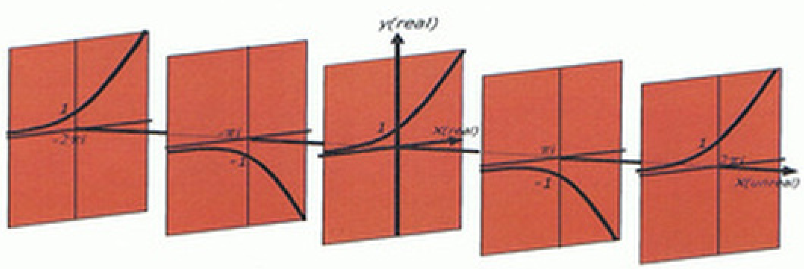
Also consider $y = e^X$ where $X = x + (2n+1)\pi i$ (ie odd numbers of π)

ie $y = e^{x + (2n+1)\pi i} = e^x \times e^{(2n+1)\pi i} = e^x \times {}^{-}1 = -e^x$

This means that the graph of $y = e^X$ consists of **parallel identical curves** if $X = x + 2n\pi i$

$= x$ + even N^{os} of πi

and, **upside down parallel identical curves** occurring at $X = x + (2n + 1)\pi i = x$ + odd N^{os} of πi



Graph of $y = e^X$ where $X = x + n\pi i$