SOME CALCULUS PRACTICE: ARCHIMEDES' CENTER OF GRAVITY OF A HEMISPHERE

William C. Evans ***July 2019***

In Archimedes' "Method", Proposition 6, he finds the center of gravity (CG) of a uniform-density hemisphere to be on its central axis, and that straight line is divided by the location of the CG in such a way that *"the portion of it adjacent to the surface of the hemisphere has to the remaining portion the ratio which 5 has to 3."* So, in other words, the CG is on the centerline, 3/8 of the way up from the flat surface of the hemisphere. He proceeds to prove this in his usual geometer's manner; let us see if we can derive this result using calculus.

Taking the vertical direction to be z, we have for the CG z-coordinate, for a hemisphere of radius R

$$\overline{z} = \frac{\int_0^R z\ \pi x_z^2\,dz}{\int_0^R \pi x_z^2\,dz}$$

with the assumption of a uniform density, which then comes outside both integrals, and cancels. (This formulation for the CG coordinates can be found in any calculus textbook.) The denominator is just the volume of the hemisphere, which of course we already know, but we will derive it anyway, for practice. These integrals use a differential volume of a disk, extending parallel to the x-y plane. By symmetry we can use either x or y (as a function of z) for the disk's radius, which decreases as we go up in the z direction from the flat surface to the top of the hemisphere. This leads to

$$\overline{z} = \frac{\int_0^R z\,\pi\left(R^2 - z^2\right)dz}{\int_0^R \pi\left(R^2 - z^2\right)dz} = \frac{\int_0^R \left(zR^2 - z^3\right)dz}{R^3 - \frac{R^3}{3}} = \frac{\frac{R^2}{2}R^2 - \frac{R^4}{4}}{\frac{2}{3}R^3} = \frac{\frac{R^4}{4}}{\frac{2}{3}R^3} = \frac{3}{8}R$$

just as Archimedes said, a very long time ago.

It might seem that a simpler way to evaluate the CG of the hemisphere would be to take a 2D slice through it at its maximum point, and find the CG of the resulting semicircle. The integration above turned out to be relatively simple, but we wouldn't know that *before* setting up the problem. The CG of the semicircle, using y instead of z to distinguish the 2D vs. 3D cases, is found using

$$\overline{y} = \frac{\int_0^R y\,x_y\,dy}{\int_0^R x_y\,dy} = \frac{\int_0^R y\sqrt{R^2 - y^2}\ dy}{\int_0^R \sqrt{R^2 - y^2}\ dy}$$

where we now have a horizontal rectangle as a differential area (so we need x as a function of y). Actually we use only the first-quadrant quarter-circle since we are using only the positive square root. Given a uniform density, this has no effect on the y-coordinate of the CG. Consider the numerator integral first; it can be written

$$\int_0^R \sqrt{R^2 - y^2}\ y\,dy$$

and if we just make a simple substitution

$$u = R^2 - y^2 \quad \Rightarrow \quad du = -2\,y\,dy$$

we will have a new integral

$$-\frac{1}{2}\int_a^b u^{\frac{1}{2}}\ du$$

We had to put the negative one-half in front of the integral to cancel the negative-2 we needed inside the integral to complete the *u*-differential. Note that the integration limits *a, b* still need to be defined, via our change of variable. Thus, using the definition of *u*, we have

$$y = 0, \quad a = R^2$$
$$y = R, \quad b = 0$$

so that the *u*-integration becomes

$$-\frac{1}{2}\int_{R^2}^{0} u^{\frac{1}{2}}\, du = \frac{1}{2}\int_{0}^{R^2} u^{\frac{1}{2}}\, du = \frac{1}{2}u^{\frac{3}{2}}\left(\frac{2}{3}\right)\Bigg|_0^{R^2} = \frac{R^3}{3}$$

Next, the denominator integral will lead to a result we already know, but it shows some interesting integration aspects that will be good practice. We have

$$\int_0^R \sqrt{R^2 - y^2}\; dy \quad \Rightarrow \quad y \equiv R\sin\theta \quad \Rightarrow \quad dy = R\cos\theta\, d\theta$$

using a trig substitution, which will lead to

$$\int_\alpha^\beta \sqrt{R^2 - R^2\sin^2\theta}\; R\;\cos\theta\; d\theta \;=\; R^2\int_\alpha^\beta \cos^2\theta\; d\theta$$

where again we must change the integration limits, based on the substitution. This gives

$$y = 0, \quad \alpha = 0$$
$$y = R, \;\; \beta = \frac{\pi}{2}$$

which we will use later. For now we need to evaluate the "anti-derivative" (olden times called this an "indefinite integral")

$$\int \cos^2\theta\, d\theta$$

which we could look up in a table of integrals. Doing so produces a result that looks like it was derived by using integration by parts. So let us try that method, with what seem to be the "obvious" choices:

$$u = \cos\theta \qquad dv = \cos\theta\, d\theta$$
$$du = -\sin\theta\, d\theta \qquad v = \sin\theta$$

This leads to

$$\sin\theta\cos\theta \;-\; \int \sin\theta\left(-\sin\theta\, d\theta\right) \;=\; \sin\theta\cos\theta \;+\; \int \sin^2\theta\; d\theta$$

and so we need to evaluate *another* integration by parts; this leads, by the same procedure as above, to

$$\int \sin^2\theta\, d\theta \;=\; -\sin\theta\cos\theta \;-\; \int -\cos^2\theta\, d\theta$$

and we have arrived at the following dead end:

$$\int \cos^2\theta\, d\theta = \sin\theta\cos\theta \;+\; \left[-\sin\theta\cos\theta + \int \cos^2\theta\, d\theta\right]$$

The integral we are attempting to find appears on both sides of the equation, and *with a positive sign on both sides*. This means that our initial choice of *u* and *dv* does not get us anywhere. This happens sometimes. So we need another approach.

Let us re-write the integral in question a bit differently:

$$\int \cos^2\theta \, d\theta \;=\; \int \left(1 - \sin^2\theta\right) d\theta$$

which may not seem like it is making any progress, since we just substitute one trig-function-squared for another. But, we have already evaluated the sine-squared integral, above, so we can write immediately

$$\int \cos^2\theta \, d\theta \;=\; \theta - \int \sin^2\theta \, d\theta \;=\; \theta - \left(-\sin\theta\cos\theta + \int \cos^2\theta \, d\theta\right)$$

and at a glance this appears to be another dead end. This time, however, we have the integral in question on the RHS with a *negative* sign, so that we can rearrange things to get

$$2\int \cos^2\theta \, d\theta \;=\; \theta + \sin\theta\cos\theta \quad\Rightarrow\quad \int \cos^2\theta \, d\theta \;=\; \frac{\theta}{2} + \frac{\sin 2\theta}{4}$$

With this, now is the time to go back to a definite integral, using the integration limits we found above. This yields

$$\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta \;=\; \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}} \;=\; \frac{\pi}{4}$$

So, as we already knew, the area of the first-quadrant circular sector is $\pi R^2 / 4$. Then the y-coordinate of the 2D CG will be

$$\bar{y} = \frac{\frac{R^3}{3}}{\frac{\pi R^2}{4}} = \frac{4}{3\pi} R$$

This plainly is *not* Archimedes' result. Thus, using the 2D slice through the 3D hemisphere will not work as a way of finding the CG of the hemisphere. Intuitively, this is because we only took a 2D slice at the *center* of the hemisphere; as we take *other* slices farther from the centerline, the CGs of those 2D slices will be lower and lower, pulling the overall CG downward. The correct hemisphere CG is at $0.375R$, while the 2D on-centerline CG is at $0.424R$.

Not only is the 2D result incorrect, that integration turned out to be *much* more difficult than doing it the right way!