A Mere Coincidence

Dr. Leon Bankoff

The Arbelos, or the Shoemaker's Knife of Archimedes, is a curious geometrical figure that is chock-full of amazing coincidences. Many of the properties presented here seem contrary to intuition; yet all can be demonstrated mathematically.

The Shoemaker's Knife is the shaded area bounded by the three semicircles in Figure 1. Each arc touches two others, and the points of tangency, A, C and B, lie on a straight line. Needless to say, the centers 0_1, 0_2 and 0 lie on the same line. In constructing the figure, the point C may be chosen anywhere on line AB.

FIG. 1

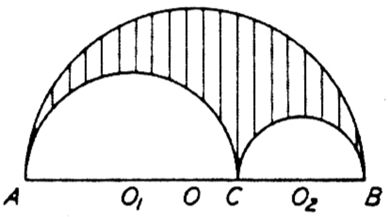
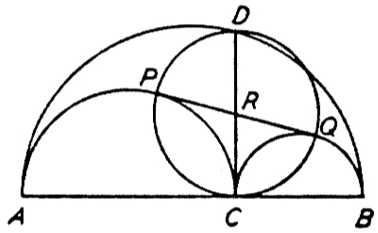


FIG. 2



In Figure 2, CD is perpendicular to AB; PQ, the common external tangent of the two smaller semicircles, cuts CD in R.

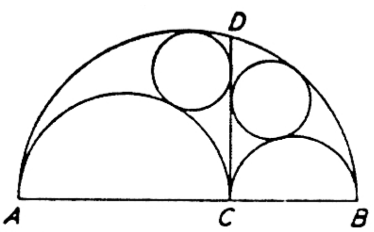
SURPRISE NO. 1: PQ = DC.

SURPRISE NO. 2: PR = CR = QR = DR. Hence, with R as center, a circle may be drawn passing through P, C, Q and D.

SURPRISE NO. 3: The area of circle PCQD is equal to the area of the arbelos. (Shaded area in Figure 1.)

Now let us consider an arbelos divided by the line CD into two unequal segments. Inscribe circles touching the boundaries of each space. (Figure 3.)

FIG. 3



SURPRISE NO. 4: The two inscribed circles are equal!

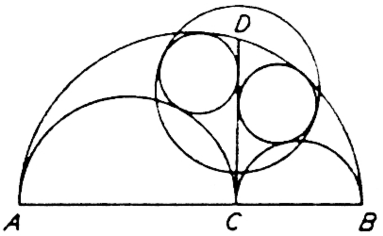


FIG. 4

SURPRISE NO. 5: The smallest circle that can enclose the two inscribed circles has a diameter equal to CD. Hence the area of this circumscribed circle is also equal to the area of the arbelos! (Figure 4.)

We now consider an arbelos having only one inscribed circle, touching arcs AC, CB and AB in P, Q and R, respectively. Let M be the mid-point of arc AC and N the mid-point of arc CB. With M as center and MA as radius, describe a circle. (Figure 5.)

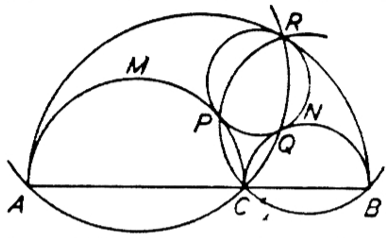


FIG. 5

SURPRISE NO. 6: The circumference of this circle will pass through C, Q and R. Similarly, the circle with center N and radius NB will pass through, C, P and R!

SURPRISE NO. 7: The perpendicular from the center of circle PQR upon the line AB is bisected by the circumference of PQR.

SURPRISE NO. 8: Draw AR, CR and BR. Angle ARB is bisected by RC.

If an arbelos is reflected in AB, as in a mirror, we obtain two congruent figures. (Figure 6.) In the lower arbelos draw CD and two inscribed circles, as in Figure 3. In the upper arbelos inscribe a circle as in Figure 5, touching the smaller semicircles in P and Q. Now draw a circle through P, C and Q.

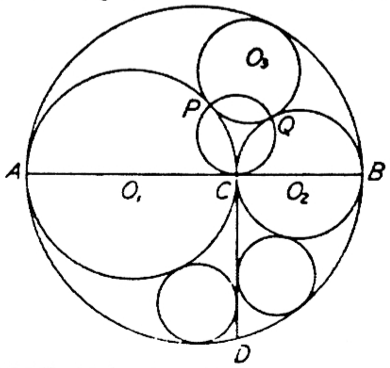


FIG. 6

SURPRISE NO. 9: The three small circles are equal!

How many of these "coincidences" can you prove? *WARNING:* Most of the proofs are not easy.

Reprinted from *Mathematics Newsletter*, Los Angeles City College, November 1954.

For more on this subject see Martin Gardner's column in the January 1979 *Scientific American* and Bankoff's "Are the twin circles of Archimedes really twins?," *Mathematics Magazine* 47 (1974) 214–218.