PROJECTILE MOTION *Velocity*

We seek to explore the velocity of the projectile, including its final value as it hits the ground, or a target above the ground. The angle made by the velocity vector with the local horizontal is also of interest. We start with the kinematic relations

$$v_x(t) = v_0 \cos(\theta) \qquad v_y(t) = v_0 \sin(\theta) - g\,t$$

and the resultant vector velocity magnitude is

$$\sqrt{\left(v_0 \cos(\theta)\right)^2 + \left(v_0 \sin(\theta) - g\,t\right)^2}$$

which can be written as

$$v(t) := \sqrt{g^2 \left(t - \frac{v_0 \sin(\theta)}{g}\right)^2 + \left(v_0 \cos(\theta)\right)^2} \tag{1}$$

where the constant in the time-parentheses is the time to attain the maximum y (vertex), as is shown in another paper in this series. So at the vertex the time component is zero and the velocity is just the x velocity. Thus the y-component of the velocity is zero at the vertex; this can also be seen with

$$v_0 \sin(\theta) - g\,t_{vertex} = v_0 \sin(\theta) - g\,\frac{v_0 \sin(\theta)}{g} = 0 = v_y$$

The angle α of the velocity vector, with respect to the local horizontal, positive counterclockwise, at any time, is, from consideration of the vector triangle

$$\alpha(t) = \operatorname{atan}\left(\frac{v_y(t)}{v_x(t)}\right) = \operatorname{atan}\left(\frac{v_0 \sin(\theta) - g\,t}{v_0 \cos(\theta)}\right) \qquad \alpha(t) := \operatorname{atan}\left(\tan(\theta) - \frac{g\,t}{v_0 \cos(\theta)}\right) \tag{2}$$

Again consider the value at the vertex:

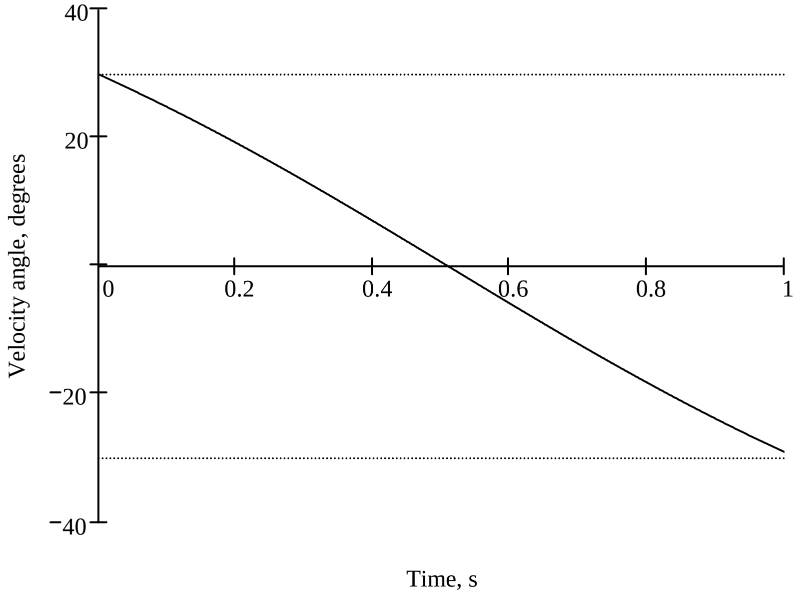
$$\alpha\left(t_{vertex}\right) = \operatorname{atan}\left(\tan(\theta) - g\,\frac{v_0 \sin(\theta)}{g\,v_0 \cos(\theta)}\right) = 0$$

Thus the velocity vector is horizontal at the vertex, so all the motion is in the x-direction. As time increases past the vertex point, the angle becomes negative, so that the motion is, as we already know, toward the ground.

Note that up to now the initial height has not entered the equations. If we wish to plot the angle as a function of time we need to consider that the flight has a finite duration- time does not increase without limit in the physical situation. We have, from another paper, the general time of flight (TOF; time from launch until the projectile reaches the ground):

$$T := \frac{1}{g}\left[v_0 \sin(\theta) + \sqrt{\left(v_0 \sin(\theta)\right)^2 + 2\,g\,y_0}\right]$$

Now we can plot the time-dependent angle. The initial height is zero, the initial velocity is 10 m/s at an angle of 30 degrees. We see that the angle is positive, decreases through zero at the vertex (which, for a zero initial height, occurs at half the TOF), and then becomes negative until it reaches the same magnitude it had at the start (more on this below). If the initial height is nonzero, this is not the case.



We can also develop this angle of the velocity vector using calculus. Starting with the parametric equations of motion,

$$x(t) = v_0 \cos(\theta)\, t \qquad\qquad y(t) = y_0 + v_0 \sin(\theta)\, t - \frac{1}{2} g\, t^2$$

the (time-dependent) derivative will be the tangent to the trajectory; for parametric equations this is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{v_0 \sin(\theta) - g\, t}{v_0 \cos(\theta)} = \tan(\theta) - \frac{g\, t}{v_0 \cos(\theta)}$$

which is the argument of the arctangent in Eq(2). Note that the magnitude of the velocity was found in Eq(1), using the kinematic equations, while the angle of the velocity vector is found in cartesian coordinates, as we just did. Another way to show this is of course to use the cartesian form of the trajectory (developed in another paper in this series), namely:

$$y(x) = \frac{-g}{2\left(v_0 \cos(\theta)\right)^2} x^2 + \tan(\theta)\, x + y_0$$

$$\frac{dy}{dx} = \frac{-2\, g\, x}{2\left(v_0 \cos(\theta)\right)^2} + \tan(\theta) = \frac{-g\left(v_0 \cos(\theta)\, t\right)}{\left(v_0 \cos(\theta)\right)^2} + \tan(\theta)$$

which is the same as we just found, above, using the parametric form. We can now write the velocity vector's angle in terms of x:

$$\alpha(x) = \operatorname{atan}\left[\frac{-g\, x}{\left(v_0 \cos(\theta)\right)^2} + \tan(\theta)\right] \qquad (3)$$

Next we develop the relations for the end of the flight, just as the projectile hits the ground. At this point, the time is the TOF, so that

$$v_x(T) = v_0 \cos(\theta) \qquad v_y(T) = v_0 \sin(\theta) - g\,T = v_0 \sin(\theta) - g\left[\frac{1}{g}\left[v_0 \sin(\theta) + \sqrt{(v_0 \sin(\theta))^2 + 2\,g\,y_0}\right]\right]$$

$$v_y(T) = -\sqrt{(v_0 \sin(\theta))^2 + 2\,g\,y_0} \qquad (4)$$

The magnitude of the resultant velocity is

$$|v(T)| = \sqrt{v_x^2 + v_y^2} = \sqrt{\left[-\sqrt{(v_0 \sin(\theta))^2 + 2\,g\,y_0}\right]^2 + (v_0 \cos(\theta))^2}$$

so that the "final" velocity magnitude will be

$$v_f = \sqrt{v_0^2 + 2\,g\,y_0} \qquad (5)$$

We can also derive this from conservation of energy:

$$\frac{1}{2}\,m\,v_0^2 + m\,g\,y_0 = \frac{1}{2}\,m\,v_f^2 + m\,g\,0$$

It can be seen by inspection that this produces the same result. This function has several interesting properties. First, the final velocity does not depend on the initial angle. Second, when the initial height is zero, the initial and final velocities are equal in magnitude. Third, as the initial velocity increases, the final velocity approaches the initial, regardless of the initial height. That is,

$$\lim_{v_0 \to \infty} \frac{\sqrt{v_0^2 + 2\,g\,y_0}}{v_0} = 1$$

A plot of Eq(5) will be of interest, but first, some analytic geometry. We can write Eq(5) as

$$v_f^2 = v_0^2 + 2\,g\,y_0 \qquad v_f^2 - v_0^2 = 2\,g\,y_0 \qquad \frac{v_f^2}{2\,g\,y_0} - \frac{v_0^2}{2\,g\,y_0} = 1$$

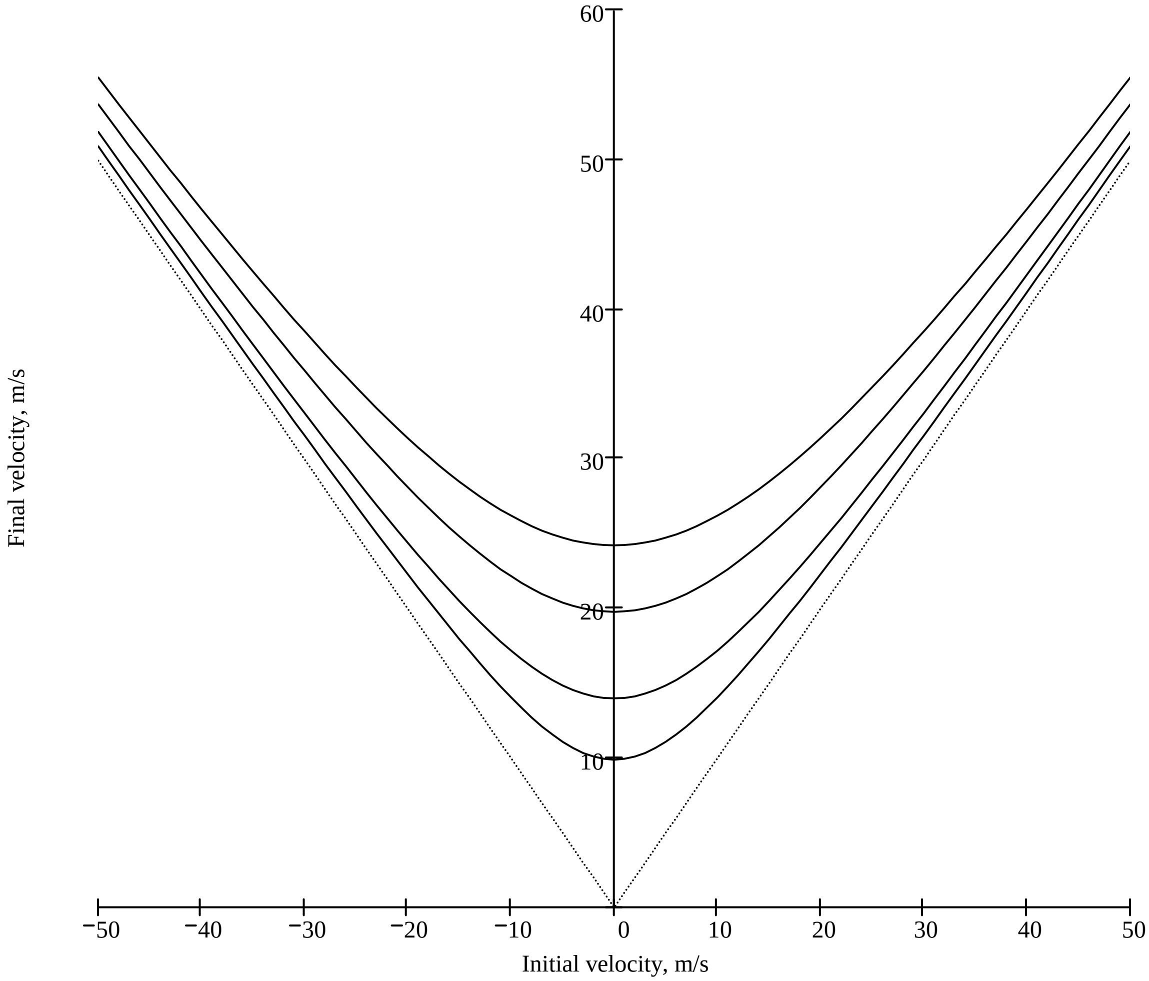
which is the form of a rectangular hyperbola:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad a = b = \sqrt{2\,g\,y_0}$$

In this case the axis of symmetry is the y-axis, and we have a "rectangular" hyperbola since a = b. The vertex is at "a" and it can be shown that the asymptotes are

$$y = \frac{a}{b}\,x \qquad y = \frac{-a}{b}\,x$$

which here will be y = x and y = -x. Next we graph this function, with the initial velocity as x and the final velocity as y.



The parameter in the plot is the initial height, at 0, 5, 10, 20, 30 m. A negative initial velocity corresponds to "throwing" the projectile downward. The dotted lines are the asymptotes, which are drawn by the zero initial height case, since then the initial and final velocities are equal. Here we have plotted the magnitude of the final velocity, but the hyperbola has a negative branch as well, and we could just as well show the final velocity as negative (since it is directed downward). A zero initial velocity means that we just drop the object from its initial height. So, e.g., from a height of 5 m, it takes about one second to reach the ground, and the velocity will then be about 10 m/s, as the plot shows (y-intercept).

A key fact, which is not intuitively obvious, is that an object projected with a given initial velocity will hit the ground with a particular final velocity (magnitude), **regardless of the initial angle**. A special case of this is to throw a ball straight up, from a bridge, as opposed to throwing it straight down from the bridge, in both cases with the same initial velocity. The ball will hit the ground with the same speed in either case. This is apparent in the symmetry of the hyperbolas in the figure. Another example would be to throw the ball at an upward angle of, say, 60 degrees, and then again at a negative angle of 40 degrees. The final velocity is the same in both cases. This is best seen from an energy viewpoint- the kinetic and potential energies are the same no matter at what angle the ball is thrown.

Also notice that the slope of the final-velocity curves is minimized for small values of the initial velocity. This can be shown with calculus, but it is apparent in the plot. The final velocity does not change much from the free-fall value (the y-intercept) when the initial velocity is small. On the other hand, when the initial velocity is large, the final velocity is essentially equal to it.

Another topic of interest is the angle of the velocity vector at impact. This would be

$$\alpha(T) = \operatorname{atan}\left(\tan(\theta) - \frac{g\ T}{v_0 \cos(\theta)}\right) \qquad \text{or} \qquad \alpha(R) = \operatorname{atan}\left[\tan(\theta) - \frac{g\ R}{\left(v_0 \cos(\theta)\right)^2}\right]$$

where R is the maximum x distance, or range. The first one is a bit easier to work with, so we have

$$\alpha(T) = \operatorname{atan}\left[\tan(\theta) + \frac{-g\left[\frac{1}{g}\left[v_0 \sin(\theta) + \sqrt{\left(v_0 \sin(\theta)\right)^2 + 2\ g\ y_0}\right]\right]}{\left(v_0 \cos(\theta)\right)^2}\right]$$

which reduces to

$$\alpha(T) = \operatorname{atan}\left[-\sqrt{\tan(\theta)^2 + \frac{2\ g\ y_0}{\left(v_0 \cos(\theta)\right)^2}}\right] \qquad (6)$$

From this we see that, if the initial height is zero, the final angle is the negative of the initial angle. Another interesting result is that, as the initial height becomes large, we find

$$\lim_{y_0 \to \infty} \operatorname{atan}\left[-\sqrt{\tan(\theta)^2 + \frac{2\ g\ y_0}{\left(v_0 \cos(\theta)\right)^2}}\right] = \frac{-\pi}{2}$$

which says that the projectile hits the ground traveling essentially straight down. Similarly, as the initial velocity becomes large, the final angle is again the negative of the initial angle, regardless of the initial height or the initial angle's value.

For completeness, we can write, using the point-slope form, an equation for the tangent line at impact (recall that y is zero at impact):

$$y_{tangent}(x) = (x - R)\left[-\sqrt{\tan(\theta)^2 + \frac{2\ g\ y_0}{\left(v_0 \cos(\theta)\right)^2}}\right] \qquad (7)$$

Finally, we can develop expressions for the velocity magnitude and angle for the case of hitting a target (at a nonzero height). The equations derived above will still be applicable, but the "TOF" will be different. For the target case, the TOF is simple:

$$T = \frac{X}{v_0 \cos(\theta)}$$

where X is the x-coordinate of the target. The angle θ needed to hit the target is specifically calculated for some given initial velocity and height (see the Targeting paper in this series). So, with this TOF, we have

$$v_x(T) = v_0 \cos(\theta) \qquad v_y(T) = v_0 \sin(\theta) - g \frac{X}{v_0 \cos(\theta)}$$

and then the velocity magnitude will be

$$|v(T)| = \sqrt{\left(v_0 \cos(\theta)\right)^2 + \left(v_0 \sin(\theta) - g \frac{X}{v_0 \cos(\theta)}\right)^2} = \sqrt{v_0^{\,2} - 2\, g\, X \tan(\theta) + \left(\frac{g\, X}{v_0 \cos(\theta)}\right)^2}$$

The velocity angle is found similarly:

$$\alpha(T) = \operatorname{atan}\left[\tan(\theta) - \frac{g\left(\dfrac{X}{v_0 \cos(\theta)}\right)}{v_0 \cos(\theta)}\right] = \operatorname{atan}\left[\tan(\theta) - \frac{g\, X}{\left(v_0 \cos(\theta)\right)^2}\right]$$