DYNAMIC RESPONSE FOR DAMPED HARMONIC OSCILLATORS

The differential equation for the generic displacement variable *x* is, from Newton's Second Law, with a linear velocity-dependent damping

$$m\left(\frac{d^2}{dt^2}x\right) + p\left(\frac{d}{dt}x\right) + qx = F\cos(\gamma\ t)$$

and, dividing by the mass we have

$$\frac{d^2}{dt^2}x + \alpha\frac{d}{dt}x + \beta\ x = \frac{F}{m}\cos(\gamma\ t) \qquad (1)$$

where α is the damping factor and β is the restoring force proportionality factor (linear restoring force with displacement). The damping is NOT FRICTION and, unlike friction, is linearly proportional to the velocity.

The solution to Eq(1) has two parts: (a) the unforced solution, aka the "free response" of the system to the initial conditions (IC), and (b) the response to the forcing function on the RHS. These solutions are linearly independent, so that the overall solution is just their sum (superposition).

UNFORCED RESPONSE

There are several ways to solve the free response portion; one useful method for our purposes is to assume an exponential response. This leads to important quantities called "eigenvalues," which here are given by

$$\lambda_1 := \frac{-\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta} \qquad \lambda_2 := \frac{-\alpha}{2} - \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta} \qquad (2)$$

We can have three situations or cases involving these eigenvalues. They can be real and distinct (different), and negative; they can be real and repeated and negative; or they can be complex conjugates with negative real parts. To begin, we define a useful quantity called the *damping ratio*

$$\zeta := \frac{\alpha}{2\sqrt{\beta}} \qquad (3)$$

CASE I Eigenvalues real, distinct, negative OVERDAMPED no oscillation $\zeta > 1$

The solution in this case follows from Eq(2) when β is **smaller** than the square of $\alpha/2$. In this case the damping "overwhelms" the restoring force and the mass does not oscillate; essentially, it just slows down and stops, and may not even return to the equilibrium position in a finite time. The solution can be shown to be, using the ICs for displacement x_0 and velocity v_0

$$x1(t) := \frac{(v_0 - \lambda_2 x_0)}{\lambda_1 - \lambda_2}\exp(\lambda_1\ t) - \frac{v_0 - \lambda_1 x_0}{\lambda_1 - \lambda_2}\exp(\lambda_2\ t) \qquad (4)$$

CASE II Eigenvalues real, repeated, negative CRITICAL DAMPING no oscillation $\zeta = 1$

In this case the mass just returns to the equilibrium position, with no overshoot. The solution is

$$x2(t) := \left[(v_0 - \lambda_1 x_0)\ t + x_0\right]\exp(\lambda_1\ t) \qquad (5)$$

Note the product of time and the exponential; this is bounded since the exponential approaches zero faster than the linear increase of time. The two eigenvalues are exactly equal in this case.

CASE III Eigenvalues complex conjugates   UNDERDAMPING   oscillation $\zeta < 1$

This is the most interesting case, since we have an exponentially-decreasing sinusoidal oscillation. This solution is a bit cumbersome, so we define some useful quantities first:

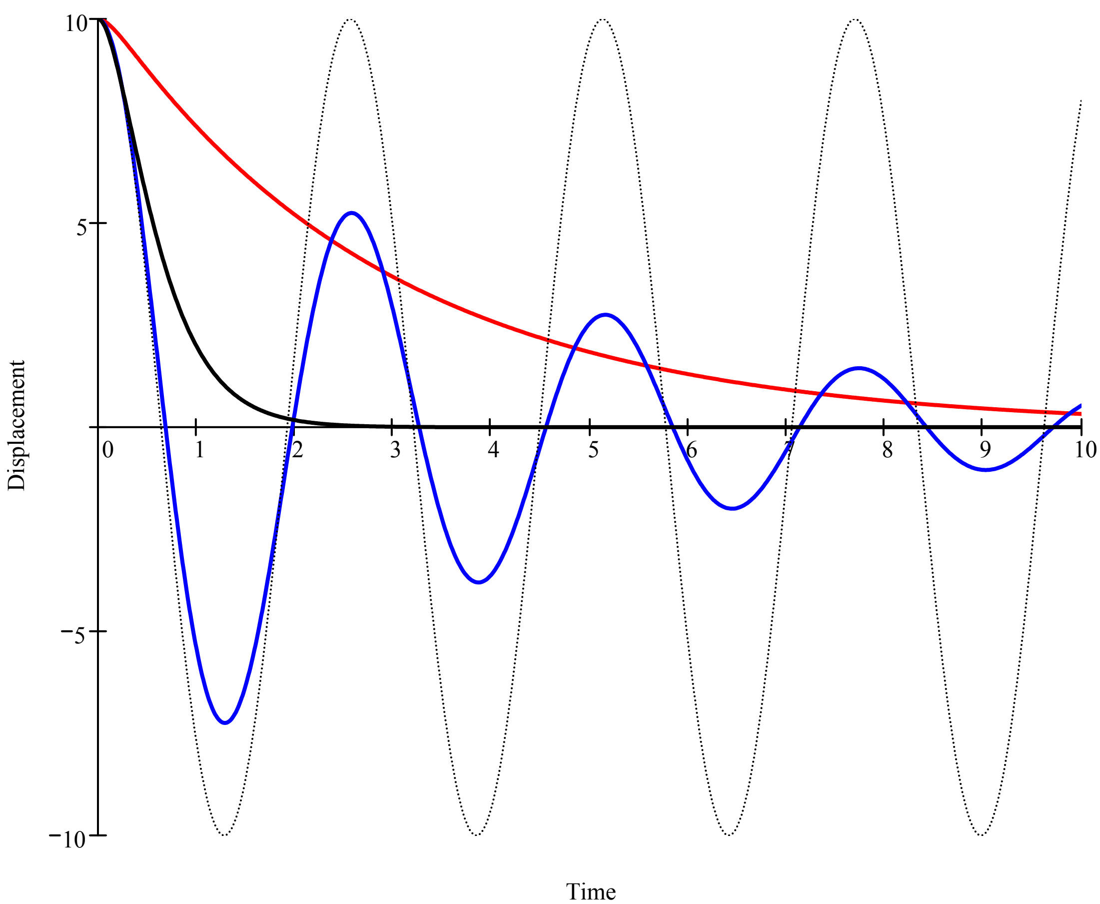
$$\omega := \sqrt{\beta - \left(\frac{\alpha}{2}\right)^2} \qquad A := \sqrt{x_0^{\,2} + \left(\frac{v_0 + x_0 \frac{\alpha}{2}}{\omega}\right)^2} \qquad \phi := \operatorname{atan}\left(\frac{v_0 + x_0 \frac{\alpha}{2}}{\omega\, x_0}\right)$$

angular frequency   amplitude   phase angle

$$x3(t) := A \exp\left(\frac{-\alpha}{2} t\right) \cos(\omega t - \phi) \tag{6}$$

This is one way to write the solution, using a single trig function with a phase angle. The more basic solution is the sum of a sine and cosine term, but combining them is more convenient for analysis of the system.

Here is a plot showing the three cases. The dotted line is Case III with no damping.



FORCED RESPONSE

We now have the predicted response of the system to its initial conditions. Next we need to develop a solution for the situation where the system is "driven" by some input, usually assumed to be a sinusoid. Again there are various methods for this; one way is to assume a solution of the form

$$x_F(t) = a_F \cos(\gamma\, t) + b_F \sin(\gamma\, t)$$

where γ is the angular frequency of the driving force, as in Eq(1). If we differentiate this twice and use those results in Eq(1), it is possible to solve for the coefficients a_F and b_F. Carrying this out we find

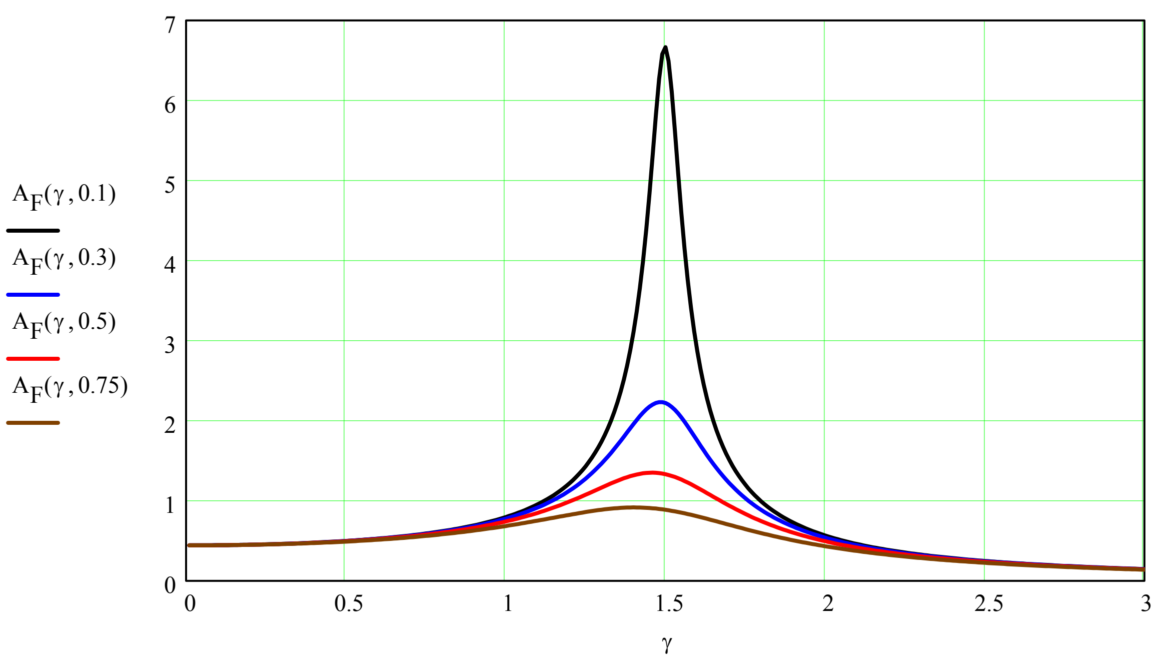
$$a_F := \frac{F_0\left(\beta - \gamma^2\right)}{\left(\gamma^2 - \beta\right)^2 + (\alpha\,\gamma)^2} \qquad\qquad b_F := \frac{F_0\,(\alpha\,\gamma)}{\left(\gamma^2 - \beta\right)^2 + (\alpha\,\gamma)^2} \tag{7}$$

Here F_0 is $F\,/\,m$. As we did for the unforced situation, it is again convenient to combine these into a single trig function, with an amplitude and phase angle. This results in

$$A_F(\gamma,\alpha) := \frac{F_0}{\sqrt{\left(\beta - \gamma^2\right)^2 + \gamma^2\,\alpha^2}} \qquad\qquad \phi_F(\alpha,\beta,\gamma) := \begin{cases} \operatorname{atan}\left(\dfrac{\alpha\,\gamma}{\beta - \gamma^2}\right) & \text{if } \left(\beta - \gamma^2\right) > 0 \\ \operatorname{atan}\left(\dfrac{\alpha\,\gamma}{\beta - \gamma^2}\right) + \pi & \text{if } \left(\beta - \gamma^2\right) < 0 \end{cases}$$

$$x_F(t) = A_F \cos\left(\gamma\, t - \phi_F\right) \tag{8}$$

Note that the phase angle must be in the first or second quadrant (i.e., between 0 and π); thus the extra complexity is needed to get the ATAN function to return the correct angle. Below is a plot of the amplitude for several values of α for F_0 = 1 and β = 1.3; the curve shifts to the left as α increases.



The peak in the amplitude graph is attained when the driving frequency γ is the same as the resonant frequency of the system, which can be shown by differentiating the equation for A_F to be

$$\omega_R = \sqrt{\beta - \frac{\alpha^2}{2}} \tag{9}$$

This situation is called **resonance,** and is of great importance in the design of mechanical systems. We can see in Eq(9) that, if the damping is large enough, the resonant frequency becomes imaginary, and, in other words, there is no resonance. In many mechanical systems this is desirable.

The maximum amplitude of the response at resonance is

$$A_{max} := \frac{F_0}{\alpha\sqrt{\beta - \left(\frac{\alpha}{2}\right)^2}} \quad \text{for small damping this is approximately} \quad A_{max} := \frac{F_0}{\alpha\sqrt{\beta}} \tag{10}$$

Note that if there is zero damping the response at resonance is, in theory, infinite.

COMBINED RESPONSE

A simulation program has been developed that implements these calculations, and permits examination of the response as the parameters are adjusted. Below is a plot showing the kind of information available in the interactive program. The thick line is the overall, total response, while the dotted line is the unforced response, and the thin solid line is the forced response.