INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

MAA

EXERCISES [MAA 5.4]

TANGENT AND NORMAL LINES

Compiled by Christos Nikolaidis

O. Practice questions

1. [Maximum mark: 16] ***[without GDC]***

Let $f(x)=2x^2-12x+10$.

(a) Find $f'(x)$ [2]

(b) Find the equations of the tangent line and the normal line

(i) at $x=1$ (ii) at $x=2$ (iii) at $x=3$ [14]

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2. [Maximum mark: 8] ***[without GDC]***

Let $f(x) = 2x^2 - 12x + 10$.

(a) Find the tangent line which is parallel to the line $y = 4x - 7$ [4]

(b) Find the tangent line which is perpendicular to the line $y = \frac{1}{4}x - 7$ [4]

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3. [Maximum mark: 7] ***[without GDC]***

Let $f(x) = e^x \cos x$.

(a) Find the gradient of the normal to the curve of f at $x = \pi$. [5]

(b) Find the gradient of the tangent to the curve of f at $x = \frac{\pi}{4}$. [2]

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4. [Maximum mark: 6] ***[without GDC]***

The line $y = mx - 25$ is tangent to the curve $f(x) = x^2$. Find the possible values of m.

METHOD A: Using derivatives (at the point of contact, $f = y$ and $f' = y'$)

METHOD B: Using $\Delta = 0$ ($f = y$ gives a quadratic equation)

5*. [Maximum mark: 8] ***[without GDC]***

The line $y = mx - 48$ is tangent to the curve $y = x^4$.

(a) Find the possible values of m. [6]

(b) Hence find the equations of the possible tangent lines. [2]

6.** [Maximum mark: 3] ***[without GDC]***

Let $y = x^4$. Find the equations of the tangent lines **passing through** the point A(0,–48).

[Notice that the point A does not lie on the line]

METHOD A: Find the equation of the line of slope m passing though A.
Then use the fact that this line is tangent to the curve to find m.

METHOD B: Find the general equation of the tangent line at any point $x = a$.
Then use the fact that it passes though A to find a.

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A. Exam style questions (SHORT)

7. [Maximum mark: 5] ***[without GDC]***

Let $f(x)=5x^2+10$. Find the equation of the tangent line at point P(1,15).

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8. [Maximum mark: 6] ***[without GDC]***

Find the equation of the tangent line and the equation of the normal to the curve with equation $y=x^3+1$ at the point (1,2).

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9. [Maximum mark: 6] ***[without GDC]***

Consider the function $f(x)=4x^3+2x$. Find the equation of the normal to the curve of f at the point where $x=1$.

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10. [Maximum mark: 10] ***[without GDC]***

Find the equations of the tangent line and the normal line to curve $y=(x-1)^4$

(a) at point P(0,1). [6]

((b) at $x=1$ [4]

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11. [Maximum mark: 8] ***[with GDC]***

Consider the curve $y = \ln(3x - 1)$. Let P be the point on the curve where $x = 2$.

(a) Write down the gradient of the curve at P. [2]

(b) Find the equation of the tangent to the curve at P. [2]

(c) The normal to the curve at P cuts the x-axis at R. Find the coordinates of R. [4]

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12. [Maximum mark: 4] ***[with GDC]***

Let $f(x) = \dfrac{x^3 e^x \ln x}{\sqrt{x+1}}$. Find the equations of the tangent line and the normal line to the curve at $x = 1$. Express both equations in the form $y = mx + c$

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13. [Maximum mark: 6] ***[with / without GDC]***

Let f be a function defined for $x > -\frac{1}{3}$ by $f(x) = \ln(3x+1)$.

(a) Find $f'(x)$.

(b) Find the equation of the normal to the curve $y = f(x)$ at the point where $x = 2$. Give your answer in the form $y = ax + b$ where $a,\ b \in \mathbb{R}$.

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14. [Maximum mark: 5] ***[without GDC]***

Consider the function $h(x) = x^{\frac{1}{5}}$.

(a) Find the equation of the tangent to the graph of h at the point where $x = a$, ($a \neq 0$). Write the equation in the form $y = mx + c$.

(b) Show that this tangent intersects the x-axis at the point $(-4a, 0)$.

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15. [Maximum mark: 4] ***[without GDC]***

Find the coordinates of the point on the graph of $y = x^2 - x$ at which the tangent is parallel to the line $y = 5x$.

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16. [Maximum mark: 6] ***[without GDC]***

Let $f(x) = kx^4$. The point $\mathrm{P}(1, k)$ lies on the curve of f. At P, the normal to the curve is parallel to $y = -\frac{1}{8}x$. Find the value of k.

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17. [Maximum mark: 6] ***[without GDC]***

Consider the function $f : x \mapsto 3x^2 - 5x + k$.

The equation of the tangent to the graph of f at $x = p$ is $y = 7x - 9$.

(a) Write down $f'(x)$.

(b) Find the value of (i) p; (ii) k.

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18. [Maximum mark: 8] ***[with / without GDC]***

Consider the curve with equation $f(x) = px^2 + qx$, where p and q are constants.

The point A(1, 3) lies on the curve. The tangent to the curve at A has gradient 8.

(a) Find the value of p and of q. [5]

(b) Find the equations of the tangent line and the normal at $x = 0.2$ [3]

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19. [Maximum mark: 8] ***[with / without GDC]***

Let $f(x) = 3\cos 2x + \sin^2 x$.

(a) Show that $f'(x) = -5\sin 2x$. [4]

(b) Find the equation of the tangent line to the graph of f at $x = 0$. [2]

(c) In the interval $\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$, one normal to the graph of f has equation $x = k$.

Find the value of k. [2]

20*. [Maximum mark: 6] ***[with GDC / without GDC for HL]***

Consider the tangent to the curve $y = x^3 + 4x^2 + x - 6$

(a) Find the equation of this tangent at the point where $x = -1$.

(b) Find the coordinates of the point where this tangent meets the curve again.

21. [Maximum mark: 6] ***[without GDC]***

The line $y = 16x - 9$ is a tangent to the curve $y = 2x^3 + ax^2 + bx - 9$ at the point (1,7).

Find the values of a and b.

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22. [Maximum mark: 6] ***[without GDC]***

The normal to the curve $y = \frac{k}{x} + \ln x^2$, for $x \neq 0$, $k \in \mathbb{R}$, at the point where $x = 2$, has equation $3x + 2y = b$, where $b \in \mathbb{R}$. Find the **exact** value of k.

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23. [Maximum mark: 6] ***[without GDC]***

Let $f(x) = 3x^2 - x + 4$. Find the values of *m* for which the line $y = mx + 1$ is a tangent to the graph of f.

METHOD A: Using quadratics and the discriminant Δ

METHOD B: Using derivatives

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24. [Maximum mark: 6] ***[without GDC]***

For what values of m is the line $y = mx + 5$ a tangent to the parabola $y = 4 - x^2$?

METHOD A: Using derivatives

METHOD B: Using quadratics and the discriminant Δ

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25.** [Maximum mark: 7] ***[without GDC]***

Let $f(x)=5x^2+10$. Find the equations of the tangent lines <u>passing through</u> the point Q(1,10).

METHOD A: Find the equation of the line of slope m passing though Q.

Then use the fact that this line is tangent to the curve to find m.

METHOD B: Find the equation of the tangent line at point $x=a$ in general.

Then use the fact that it passes though Q to find a.

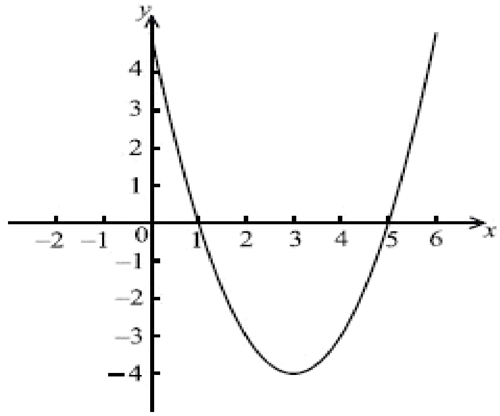
26.** [Maximum mark: 7] ***[without GDC]***

Find the equations of the two tangent lines to the parabola $y = 4 - x^2$, which pass through the point (0,5).

B. Exam style questions (LONG)

27. [Maximum mark: 10] ***[without GDC]***

The following diagram shows part of the graph of a quadratic function, with equation in the form $y=(x-p)(x-q)$, where $p,q\in\mathbb{Z}$.



(a) (i) Write down the value of p and of q

(ii) Write down the equation of the axis of symmetry of the curve.

(b) Find the equation of the function in the form $y=(x-h)^2+k$, where $h,k\in\mathbb{Z}$.

(c) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$

(d) Let *T* be the tangent to the curve at the point $(0, 5)$. Find the equation of *T*.

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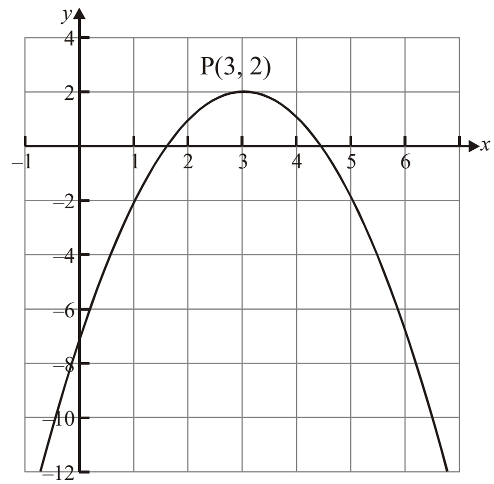
28. [Maximum mark: 13] ***[with GDC]***

The function $f(x)$ is defined as $f(x) = -(x-h)^2 + k$. The diagram below shows part of the graph of $f(x)$. The maximum point on the curve is P (3, 2).



(a) Write down the value of (i) h (ii) k [2]

(b) Show that $f(x)$ can be written as $f(x) = -x^2 + 6x - 7$. [1]

(c) Find $f'(x)$. [2]

The point Q lies on the curve and has coordinates (4, 1). A straight line *L*, through Q, is perpendicular to the tangent at Q.

(d) (i) Find the equation of *L*.

(ii) The line *L* intersects the curve again at R. Find the x-coordinate of R. [8]

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29. [Maximum mark: 11] ***[with GDC]***

The function f is defined by $f : x \mapsto -0.5x^2 + 2x + 2.5$.

(a) Write down (i) $f'(x)$; (ii) $f'(0)$. [2]

(b) Let *N* be the normal to the curve at the point where the graph intercepts the y-axis. Show that the equation of *N* may be written as $y = -0.5x + 2.5$. [3]

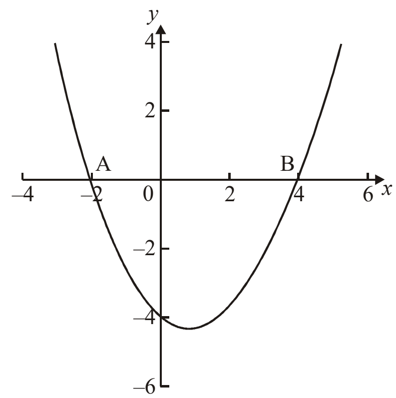
Let $g : x \mapsto -0.5x + 2.5$

(c) (i) Find the solutions of $f(x) = g(x)$

(ii) Hence find the coordinates of the other point of intersection of the normal and the curve. [6]

30. [Maximum mark: 15] ***[with / without GDC]***

The equation of a curve may be written in the form $y = a(x-p)(x-q)$. The curve intersects the x-axis at A(–2, 0) and B(4, 0). The curve of $y = f(x)$ is shown in the diagram below.



(a) (i) Write down the value of p and of q.

(ii) Given that the point (6, 8) is on the curve, find the value of a.

(iii) Write the equation of the curve in the form $y = ax^2 + bx + c$. [5]

(b) A tangent is drawn to the curve at a point P. The gradient of this tangent is 7. Find the coordinates of P. [4]

(c) The line *L* passes through B(4, 0), and is normal to the curve at B.

(i) Find the equation of *L*.

(ii) Find the x-coordinate of the point where *L* intersects the curve again. [6]

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31. [Maximum mark: 24] ***[without GDC]***

The function f is given by $f(x)=\dfrac{2x+1}{x-3}$, $x\in\mathbb{R}$, $x\neq 3$.

(a) (i) Show that $y=2$ is an asymptote of the graph of $y=f(x)$.

(ii) Find the vertical asymptote of the graph.

(iii) Write down the coordinates of the point *P* at which the asymptotes intersect. [4]

(b) Find the points of intersection of the graph and the axes. [4]

(c) Hence sketch the graph of $y=f(x)$, showing the asymptotes by dotted lines. [4]

(d) Show that $f'(x)=\dfrac{-7}{(x-3)^2}$ and hence find the equation of the tangent at the point *S* where $x=4$. [6]

(e) The tangent at the point *T* on the graph is parallel to the tangent at *S*. Find the coordinates of *T*. [5]

(f) Show that *P* is the midpoint of [*ST*]. [1]

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32. [Maximum mark: 9] ***[with GDC]***

Let $f(x) = x^3 - 3x^2 - 24x + 1$.

(a) Find $f'(x)$ [2]

The tangents to the curve of f at the points P and Q are parallel to the x-axis, where P is to the left of Q.

(b) Calculate the coordinates of P and of Q. [3]

Let N_1 and N_2 be the normals to the curve at P and Q respectively.

(b) Write down the coordinates of the points where

(i) the tangent at P intersects N_2;

(ii) the tangent at Q intersects N_1. [4]

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33. [Maximum mark: 14] ***[with GDC]***

(a) Sketch and label the curves $y = x^2$ for $-2 \le x \le 2$, and $y = -\frac{1}{2}\ln x$ for $0 < x \le 2$. [2]

(b) Find the x-coordinate of P, the point of intersection of the two curves. [2]

(c) If the tangents to the curves at P meet the y-axis at Q and R, calculate the area of the triangle PQR. [6]

(d) Prove that the two tangents at the points where $x = a$, $a > 0$, on each curve are always perpendicular. [4]

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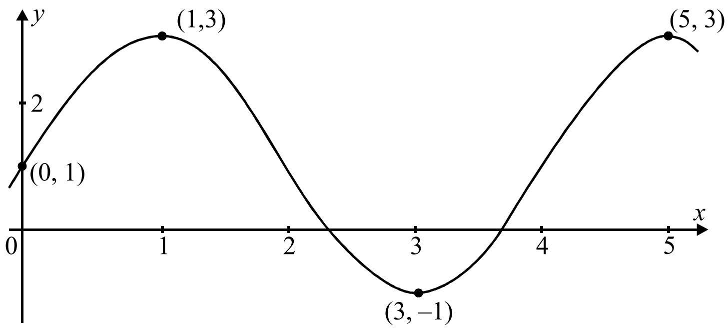
34*. [Maximum mark: 20] ***[with / without GDC]***

The diagram shows the graph of the function f given by $f(x) = A\sin\left(\frac{\pi}{2}x\right) + B$, for $0 \le x \le 5$, where A and B are constants, and x is measured in radians.



The graph includes the points (1, 3) and (5, 3), which are maximum points of the graph.

(a) Show that $A = 2$, and find the value of B. [5]

(b) Show that $f'(x) = \pi\cos\left(\frac{\pi}{2}x\right)$. [4]

The line $y = k - \pi x$ is a tangent line to the graph for $0 \le x \le 5$.

(c) Find

(i) the point where this tangent meets the curve;

(ii) the value of k. [6]

(d) Solve the equation $f(x) = 2$ for $0 \le x \le 5$. [5]

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35.** [Maximum mark: 15] ***[with GDC]***

(a) The function g is defined by $g(x)=\frac{e^x}{\sqrt{x}}$, for $0<x\le 3$.

(i) Sketch the graph of g .

(ii) Find $g'(x)$

(iii) Write down an expression representing the gradient of the normal to the curve at any point. [8]

(b) Let P be the point (x,y) on the graph of g , and Q the point $(1,0)$.

(i) Find the gradient of (PQ) in terms of x .

(ii) Given that the line (PQ) is a normal to the graph of g at the point P , find the minimum distance from the point Q to the graph of g . [7]

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