Teacher Solution to the Task

1. To solve the equation $z^3 - 1 = 0$ one could try solving by allowing $z = a + bi$ and then expanding to the following:

$$\left(a+bi\right)^3 - 1 = 0$$
$$a^3 + 3a^2bi + 3a\left(bi\right)^2 + \left(bi\right)^3 - 1 = 0$$
$$\left(a^3 - 3ab^2 - 1\right) + \left(3a^2b - b^3\right)i = 0$$

At this point, separating the problem into the real imaginary components we would have the following system of equations:

$$a^3 - 3ab^2 - 1 = 0$$
$$3a^2b - b^3 = 0$$

Solving:

$$3a^2b - b^3 = 0$$
$$b\left(3a^2 - b^2\right) = 0$$
$$b = 0, \text{ or } 3a^2 - b^2 = 0$$
$$a = \pm\tfrac{b}{\sqrt{3}}$$

$$\text{if } b = 0:$$
$$a^2 - 3a\left(0\right)^2 - 1 = 0$$
$$a^2 = 1$$
$$a = \pm 1$$
$$\therefore z = 1 + 0bi \text{ or } z = -1 + 0bi$$
$$\text{but, } z = -1 \text{ doesn't satisfy the original equation}$$

$$\text{if } a = \tfrac{b}{\sqrt{3}}$$
$$\left(\tfrac{b}{\sqrt{3}}\right)^3 - 3\left(\tfrac{b}{\sqrt{3}}\right)b^2 - 1 = 0$$
$$\tfrac{b^3}{3\sqrt{3}} - \tfrac{3b^3}{\sqrt{3}} - 1 = 0$$
$$b^3 - 9b^3 - 3\sqrt{3} = 0$$
$$-8b^3 = 3\sqrt{3}$$
$$\left(b^3\right)^{\frac{1}{3}} = \left(-\frac{\sqrt{27}}{8}\right)^{\frac{1}{3}}$$
$$b = -\tfrac{\sqrt{3}}{2} \quad \therefore a = \tfrac{-\frac{\sqrt{3}}{2}}{\sqrt{3}} = -\tfrac{1}{2}$$
$$z = -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i$$

$$\text{if } a = -\tfrac{b}{\sqrt{3}}$$
$$\left(-\tfrac{b}{\sqrt{3}}\right)^3 - 3\left(-\tfrac{b}{\sqrt{3}}\right)b^2 - 1 = 0$$
$$-\tfrac{b^3}{3\sqrt{3}} + \tfrac{3b^3}{\sqrt{3}} - 1 = 0$$
$$-b^3 + 9b^3 - 3\sqrt{3} = 0$$
$$8b^3 = 3\sqrt{3}$$
$$\left(b^3\right)^{\frac{1}{3}} = \left(\frac{\sqrt{27}}{8}\right)^{\frac{1}{3}}$$
$$b = \tfrac{\sqrt{3}}{2} \quad \therefore a = -\tfrac{\frac{\sqrt{3}}{2}}{\sqrt{3}} = -\tfrac{1}{2}$$
$$z = -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i$$

$$\therefore z = 1, -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i \text{ and } -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i$$

An easier solution to solving $z^3 - 1 = 0$ is to use **De Moivre's Theorem**:

$z^3 = 1$

$z^3 = cis(0 + 2n\pi),\ n \in \mathbb{Z}$

$\therefore z = \left[cis(0 + 2n\pi)\right]^{\frac{1}{3}}$

$z = cis\left[\tfrac{1}{3}(2n\pi)\right]$

$if\ n = -1, 0, 1$

$z = cis\left(-\tfrac{2\pi}{3}\right),\ cis(0),\ cis\left(\tfrac{2\pi}{3}\right)$

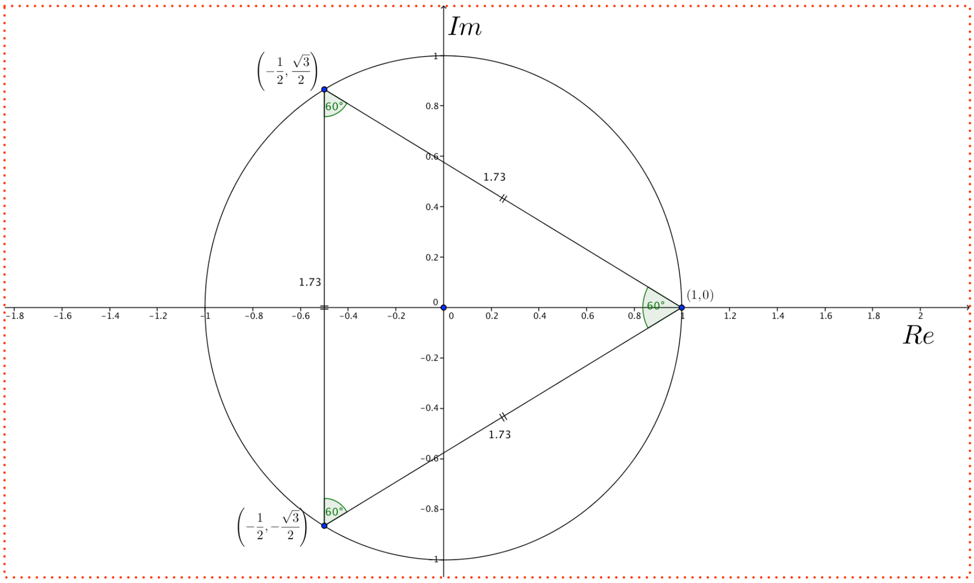
putting these solutions into cartesian coordinate form gives:

$$\begin{aligned} z &= cis\left(-\tfrac{2\pi}{3}\right) \\ &= \cos\left(-\tfrac{2\pi}{3}\right) + i\sin\left(-\tfrac{2\pi}{3}\right) \\ &= -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i \end{aligned}$$

$$\begin{aligned} z &= cis(0) \\ &= \cos(0) + i\sin(0) \\ &= 1 + 0i \end{aligned}$$

$$\begin{aligned} z &= cis\left(\tfrac{2\pi}{3}\right) \\ &= \cos\left(\tfrac{2\pi}{3}\right) + i\sin\left(\tfrac{2\pi}{3}\right) \\ &= -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i \end{aligned}$$

Placing these three solutions on the Argand plane gives the following:



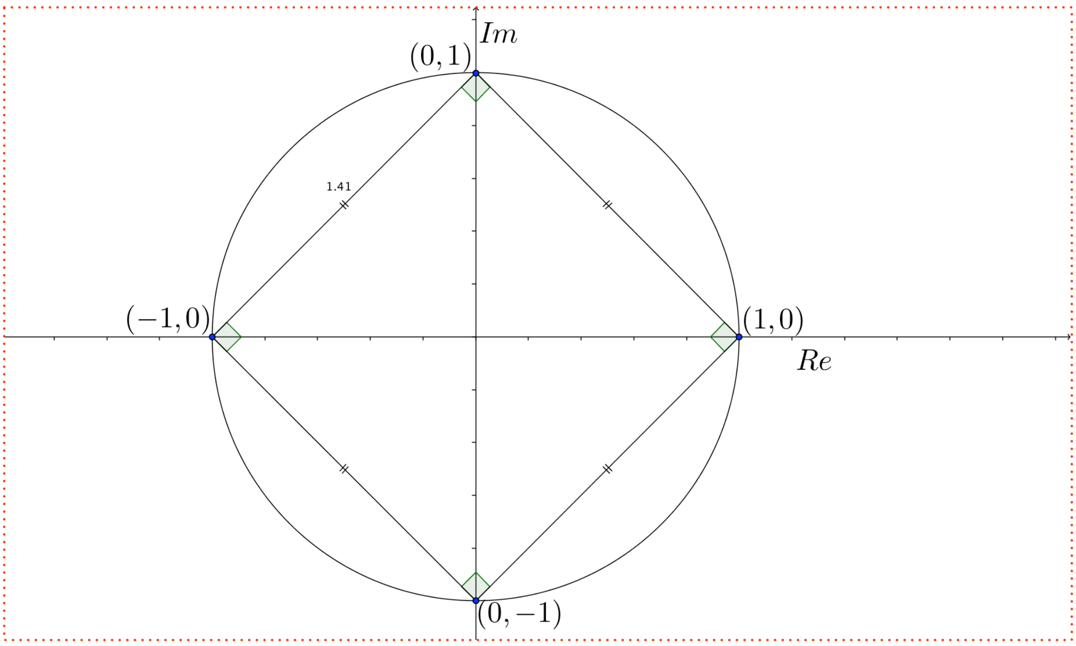
Note the following:

- the three roots are symmetrically placed such that the three complex numbers form an equilateral triangle
- the length of each segment that connects two of the complex roots is $\sqrt{3} \approx 1.73 units$
- if we draw lines from one root to the other two roost it then follows that the sum of these distances would be $\sqrt{3} + \sqrt{3} = 2\sqrt{3} units$
- if we multiply these two lengths the result is 3

2. Solving $z^4 - 1 = 0$ using De Moivre's theorem:

$$
\begin{aligned}
&z^4 = 1\\
&z^4 = cis(0 + 2n\pi),\ n \in \mathbb{Z}\\
&z = \left[cis(0 + 2n\pi)\right]^{\frac{1}{4}}\\
&z = cis\left(\tfrac{n\pi}{2}\right)\\
&so,\ for\ n = -1,0,1,2\ we\ have\\
&z = cis\left(-\tfrac{\pi}{2}\right), cis(0), cis\left(\tfrac{\pi}{2}\right), cis(\pi)\\
&in\ cartesian\ coordinate\ form: z = -i,1,i,-1
\end{aligned}
$$

Graphing the solutions on the Argand plane with a unit circle gives the following:

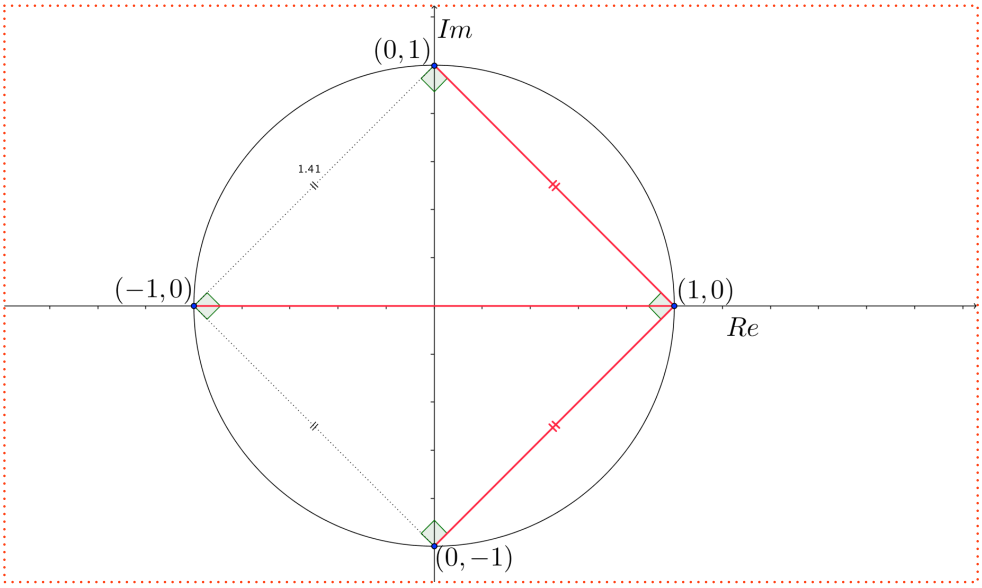


Note:

- it should be noted that an n-sided regular polygon is formed once connecting all the adjacent roots of the equation $z^n - 1 = 0$
- when $n = 4$, we have a square with side lengths of $\sqrt{2}$, an example of one of the side calculations is:

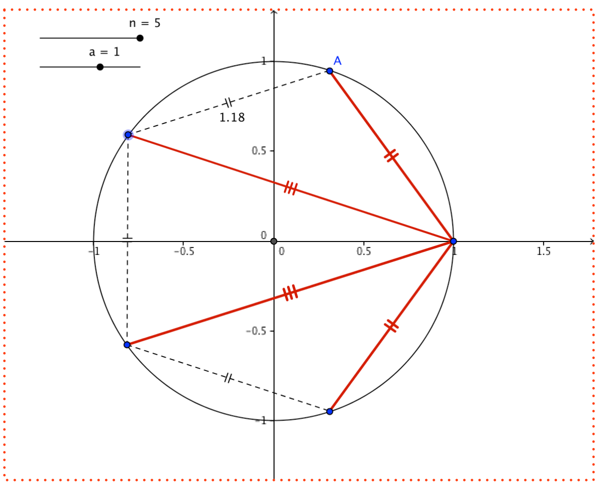
$$
\begin{aligned}
d &= \sqrt{\left(\cos(0) - \cos\left(\tfrac{\pi}{2}\right)\right)^2 + \left(\sin(0) - \sin\left(\tfrac{\pi}{2}\right)\right)^2}\\
&= \sqrt{(1-0)^2 + (0-1)^2}\\
&= \sqrt{2}
\end{aligned}
$$

- If we determine the distance from the root at $(1,0)$ to each of the other three roots we will find the total distance to be $\sqrt{2}+\sqrt{2}+2=2\sqrt{2}+2$ units.
- the product of the three distances is $\sqrt{2}\times\sqrt{2}\times 2=4$



3. Solving $z^5-1=0$ using De Moivre's

$$z^5=1$$
$$z^5=cis(0+2n\pi),\ n\in\mathbb{Z}$$
$$z=\left[cis(0+2n\pi)\right]^{\frac{1}{5}}$$
$$z=cis\left(\tfrac{2n\pi}{5}\right)$$
$$so,\ for\ n=-2,-1,0,1,2\ we\ have$$
$$z=cis\left(-\tfrac{4}{5}\pi\right),cis\left(-\tfrac{2}{5}\pi\right),cis(0),cis\left(\tfrac{2}{5}\pi\right),cis\left(\tfrac{4}{5}\pi\right)$$



Note on Technology Use

- the diagrams were done using sliders in GeoGebra. A point was defined as $\left(\cos\left(\frac{2a\pi}{n}\right),\sin\left(\frac{2a\pi}{n}\right)\right)$, where the variables n and a were adjusted for each case
- for example, when solving $z^5-1=0$, n was set to 5 and the slider a was defined as $\{a\in\mathbb{Z} \mid -2\le a\le 2\}$. Then trace was set on and the animation was turned on. The solutions were then plotted and the line segments were added in after the plotting of the solution points.
- Since the animation with the trace feature on only indicated where the roots appeared, but did not give the actual points at the roots - when the line segments were put in manually afterwards the points did not end up in the precise location.
- if I was willing to devote more time I could improve upon this process by manually defining 7 or 8 points as:
    - $A\left(\cos\left(\frac{2n\pi}{a}\right),\sin\left(\frac{2n\pi}{a}\right)\right)$
    - $B\left(\cos\left(\frac{2(n+1)\pi}{a}\right),\sin\left(\frac{2(n+1)\pi}{a}\right)\right)$
    - $C\left(\cos\left(\frac{2(n+2)\pi}{a}\right),\sin\left(\frac{2(n+2)\pi}{a}\right)\right)$

Algebraically Finding the Distances

- We only need to find two of the distances and by symmetry the other two respective side lengths will be the same.
- Therefore, we will find the distance from $(1,0)$ to the points defined as $cis\left(\frac{2\pi}{5}\right)$ and $cis\left(\frac{4\pi}{5}\right)$.

$(1,0)$ *to* $cis\left(\frac{2\pi}{5}\right)$

$$\begin{aligned} d &= \sqrt{\left(1-\cos\left(\tfrac{2\pi}{5}\right)\right)^2+\left(0-\sin\left(\tfrac{2\pi}{5}\right)\right)^2} \\ &\approx \sqrt{0.47746+0.90451} \\ &\approx \sqrt{1.381966} \\ &\approx 1.17557 \end{aligned}$$

$(1,0)$ *to* $cis\left(\frac{4\pi}{5}\right)$

$$\begin{aligned} d &= \sqrt{\left(1-\cos\left(\tfrac{4\pi}{5}\right)\right)^2+\left(0-\sin\left(\tfrac{4\pi}{5}\right)\right)^2} \\ &\approx \sqrt{3.27254+0.34549} \\ &\approx \sqrt{3.61803} \\ &\approx 1.90211 \end{aligned}$$

- the sum of the distances would then be $2(1.17557)+2(1.90211)\approx 6.15537$
- the product of these distances would be $(1.7557)^2(1.90211)^2=5$

Forming a Conjecture

- The results found for the sums of the lengths from one root to the other roots is given in the table below:

n	Distances	Sum
3	$\sqrt{3}+\sqrt{3}$	$2\sqrt{3}\approx 3.464102$
4	$\sqrt{2}+\sqrt{2}+2$	$2+2\sqrt{2}\approx 4.828427$
5	Using GeoGebra (approximate values) $1.176+1.904+1.906+1.172$	Approximately $6.158\mathit{units}$
	Using the algebraic method with a TI-84 $2(1.17557)+2(1.90211)$	Approximately $6.155\mathit{units}$

Note:

- At this point the students need to come up with the correct conjecture. I think having the students focus on the first two cases only and not focusing on the SUM may help them see the trick!
- Before allowing the students to get to far into the task, I think we will need to discourage any other patterns they come up with.
- Other patterns are there, but I think they will not tie into the factorization nor will they expand into the cases where the modulus is not one

Conjecture

- For $z^n-1=0$, if one root is isolated and then the distances are found from that one root to the other $n-1$ roots, then the product of the distances will equal n.

What about n of 2? n of 1?

- $z^2=1$ will give two real and only two real roots, but these roots will be $(-1,0)$ and $(1,0)$. These two roots are at a distance of two units away. Here, the distance between the two roots is equal to the exponent, 2.
- if we have $n=1$, then the equation $z^n=1$ becomes simply $z=1$. In this case we have no roots to find the distance to the only one root present and so this case does not fit our general statement.

Correct Conjecture

- For $z^n-1=0$, $n\in\mathbb{Z}, n\geq 2$ if one root is isolated and then the distances are found from that one root to the other $n-1$ roots, then the product of the distances will equal n.

Factorization

$z^3 - 1$

$PZV's: \pm 1$

$if\ z = 1, P(1) = 0 \therefore z - 1\ is\ a\ root$

$(z-1)(z^2 + az + b)$

$a = b = 1$

$(z-1)(z^2 + z + 1)$

$z^4 - 1$

$PZV's: \pm 1$

$if\ z = 1, P(1) = 0 \therefore z - 1\ is\ a\ factor$

$(z-1)(z^3 + az^2 + bz + c)$

$a = b = c = 1$

$(z-1)(z^3 + z^2 + z + 1)$

$(z-1)(z+1)(z^2 + az + b)$

$(z-1)(z+1)(z^2 - z + 1)$

$z^5 - 1$

$if\ z = 1, P(1) = 0 \therefore z - 1\ is\ a\ root$

$(z-1)(z^4 + az^3 + bz^2 + cz + 1)$

$a = b = c = 1$

$(z-1)(z^4 + z^3 + z^2 + z + 1)$

$$(z-1)\left(z^2 + \left(\frac{1+\sqrt{5}}{2}\right)z + 1\right)\left(z^2 + \left(\frac{1-\sqrt{5}}{2}\right)z + 1\right)$$

$z^4 + z^3 + z^2 + z + 1$

$= (z^2 + az + 1)(z^2 + bz + 1)$

$so,$

$(a+b)z^3 = 1z^3$

$(ab+2)z^2 = z^2$

$(a+b)z = z$

$\therefore a + b = 1\ and\ ab = -1$

$a = 1 - b \qquad (1-b)b + 1 = 0$

$b^2 - b - 1 = 0$

$so, b = \dfrac{1 \pm \sqrt{5}}{2}$

$this\ gives\ a = \dfrac{1 \mp \sqrt{5}}{2}$

Alternatively, since $z^5 - 1 = 0$ has roots of $1, cis\left(\pm\frac{2\pi}{5}\right)$ and $cis\left(\pm\frac{4\pi}{5}\right)$ to determine the quadratics we could multiply the conjugates. This would give us:

$$\left(z - \cos\left(\tfrac{2\pi}{5}\right) - i\sin\left(\tfrac{2\pi}{5}\right)\right)\left(z - \cos\left(-\tfrac{2\pi}{5}\right) - i\sin\left(-\tfrac{2\pi}{5}\right)\right)$$
$$= \left(z - \cos\left(\tfrac{2\pi}{5}\right) - i\sin\left(\tfrac{2\pi}{5}\right)\right)\left(z - \cos\left(\tfrac{2\pi}{5}\right) + i\sin\left(\tfrac{2\pi}{5}\right)\right),\ as\ \cos\theta = \cos(-\theta)\ and\ \sin\theta = -\sin(-\theta)$$
$$= z^2 - 2z\cos\left(\tfrac{2\pi}{5}\right) + \cos^2\left(\tfrac{2\pi}{5}\right) + \sin^2\left(\tfrac{2\pi}{5}\right)$$
$$= z^2 - 2z\cos\left(\tfrac{2\pi}{5}\right) + 1,\ since\ \cos^2\theta + \sin^2\theta = 1$$

$$\left(z-\cos\left(\tfrac{4\pi}{5}\right)-i\sin\left(\tfrac{4\pi}{5}\right)\right)\left(z-\cos\left(-\tfrac{4\pi}{5}\right)-i\sin\left(-\tfrac{4\pi}{5}\right)\right)$$
$$=\left(z-\cos\left(\tfrac{4\pi}{5}\right)-i\sin\left(\tfrac{4\pi}{5}\right)\right)\left(z-\cos\left(\tfrac{4\pi}{5}\right)+i\sin\left(\tfrac{4\pi}{5}\right)\right),\ as\ \cos\theta=\cos(-\theta)\ and\ \sin\theta=-\sin(-\theta)$$
$$=z^2-2z\cos\left(\tfrac{4\pi}{5}\right)+\cos^2\left(\tfrac{4\pi}{5}\right)+\sin^2\left(\tfrac{4\pi}{5}\right)$$
$$=z^2-2z\cos\left(\tfrac{4\pi}{5}\right)+1,\ since\ \cos^2\theta+\sin^2\theta=1$$

$$\therefore z^5-1=(z-1)\left(z^2-2z\cos\left(\tfrac{2\pi}{5}\right)+1\right)\left(z^2-2z\cos\left(\tfrac{4\pi}{5}\right)+1\right)$$

For the factoring it was unclear how far the students needed to go; I think linear and quadratic factors is sufficient. However, for the proof it may have been easier and better that the students simply listed the factorizations in the form:

$$(z-1)\left(z-\cos\left(-\tfrac{2\pi}{3}\right)-i\sin\left(-\tfrac{2\pi}{3}\right)\right)\left(z-\cos\left(\tfrac{2\pi}{3}\right)-i\sin\left(\tfrac{2\pi}{3}\right)\right)$$
$$=(z-1)\left(z-\cos\left(\tfrac{2\pi}{3}\right)\pm i\sin\left(\tfrac{2\pi}{3}\right)\right)$$

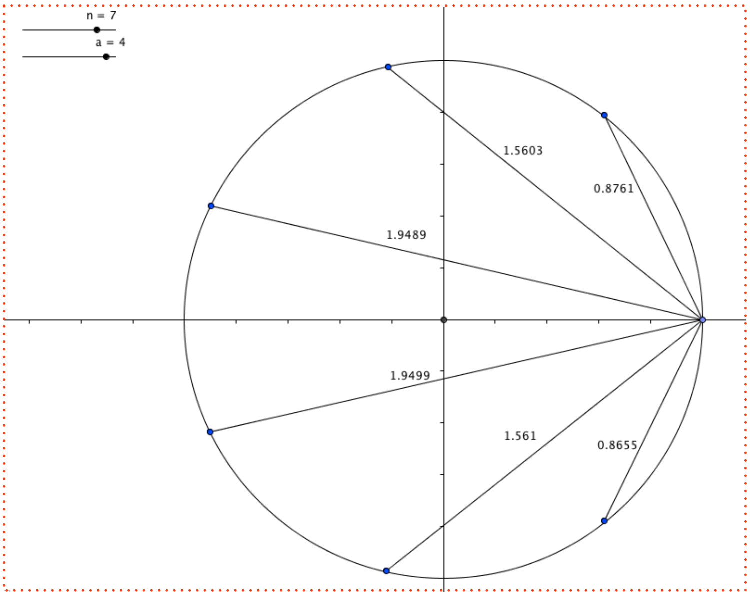
Using Technology to Test Other Cases

- for all values of n, $(1,0)$ will always be a root
- if n is odd then there will obviously be an odd number of roots, but one root will as indicated be $(1,0)$ and the other even number of roots will be in pairs that are vertical reflections of one another across the real axis.
- if n is even then there will be an even number of roots, but two of the roots will always be $(1,0)$ and $(-1,0)$. The remaining even number of roots will once again occur in pairs that are vertical reflections of one another across the real axis.
- with this in mind, as the students verified more cases using technology it was preferred that they did a at least one even and one odd value for n (two or three cases was sufficient)

GeoGebra Usage

- As explained in the $z^5-1=0$ case, sliders where utilized to plot the points
- After the points were plotted, line segments were drawn in connecting the roots to the root $(1,0)$
- For each line segment the length of the line segment to four decimal places were displayed - in some cases the corresponding pairs of distances were not always exactly the same indicating there was some degree of error within GeoGebra. However, the products for the distances would consistently be n if rounded to the nearest thousandths.

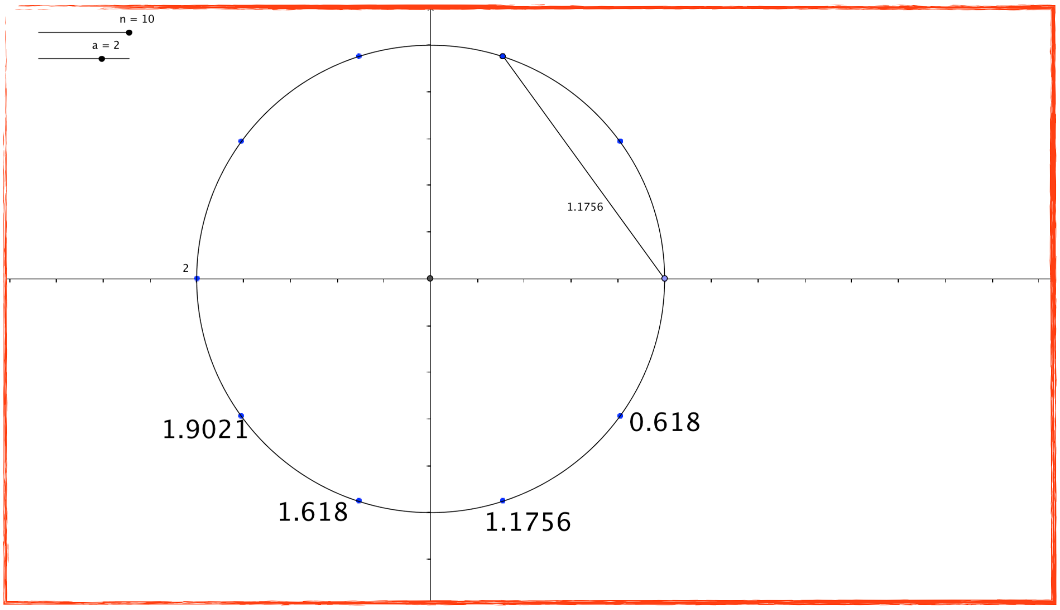
if n is 7:



Multiplying the distances above given by GeoGebra:

- $(0.8761)(1.5603)(1.9489)(1.9499)(1.561)(0.8655) \approx 7.01832$

If n is 10:



Multiplying the distances we would have:

$2(1.9021)^2(1.618)^2(1.175)^2(0.618)^2 \approx 9.98864100$

Note:

- these values could be made closer to the desired 7 and 10 by having GeoGebra show more decimals or by linking points to the sliders as explained earlier.

Proof of Our General Statement

- I think this part of the task was going to be extremely difficult and may actually advise my students that perhaps skipping this may be a strategic time management decision. A lack of a formal proof precludes the students from a perfect five out of five for the results; but the correct general statement with the correct scope and limitations is four out of five.

In looking at the factorizations it should be noted that $z^n - 1$ will always factor to:

- $(z-1)(z^{n-1}+z^{n-2}+...+z^2+z+1)$

However, the factorization of $z^n - 1$ is also:

- $(z-1)(z-w)(z-w^2)...(z-w^{n-1})$, where w is the complex root with the smallest argument

from this we have:

$$(z-1)(z^{n-1}+z^{n-2}+...+z^2+z+1)=(z-1)(z-w)(z-w^2)...(z-w^{n-1})$$
$$(z^{n-1}+z^{n-2}+...+z^2+z+1)=(z-w)(z-w^2)...(z-w^{n-1})$$

If $z=1$ then the left hand side will become n and the right hand side is $(1-w)(1-w^2)...(1-w^{n-1})$

So, from here we can deduce that their modulus' should be equal, and so:

$$|n|=\left|(1-w)(1-w^2)...(1-w^{n-1})\right|$$
$$n=\left|(1-w)(1-w^2)...(1-w^{n-1})\right|$$
$$\therefore n=|1-w|\times|1-w^2|\times...\times|1-w^{n-1}|,\ as\ |z_1z_2...z_n|=|z_1||z_2|...|z_n|$$

Note:

- the above proof is from the OCC from Ferenc Beleznay.
- I would really like to thank Ferenc for this eloquent, yet simple proof that was within my realm of understanding!!

Part B

- Solving $z^n = i$ for values of $n = 3,4,5$

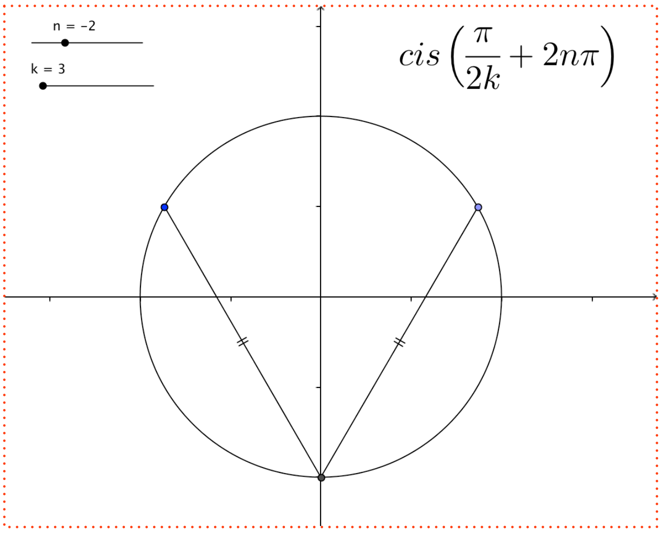
$z^3 = i$

$z^3 = cis\left(\frac{\pi}{2} + 2n\pi\right),\ n \in \mathbb{Z}$

$\therefore z = cis\left(\frac{\pi}{6} + \frac{2n\pi}{3}\right)$

$so,\ for\ n = -1,0,1$

$z = cis\left(-\frac{\pi}{2}\right), cis\left(\frac{\pi}{6}\right), cis\left(\frac{5\pi}{6}\right)$



Note:

- this is the same diagram as $z^3 = 1$, except that the three roots have all been rotated about the origin $\frac{\pi}{6}\, radians$
- so, the product of the distances in this case will also be 3, since $\sqrt{3} \times \sqrt{3} = 3$

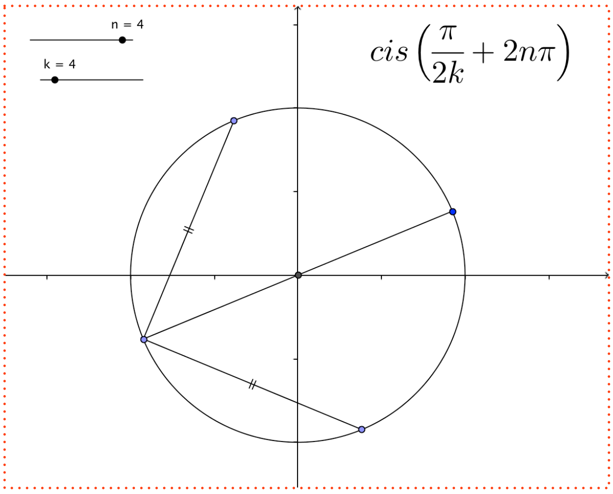
$z^4 = i$

$z^4 = cis\left(\frac{\pi}{2} + 2n\pi\right),\ n \in \mathbb{Z}$

$z = cis\left(\frac{\pi}{8} + \frac{n\pi}{2}\right)$

$so,\ for\ n = -2,-1,0,1$

$z = cis\left(-\frac{7\pi}{8}\right), cis\left(-\frac{3\pi}{8}\right), cis\left(\frac{\pi}{8}\right), cis\left(\frac{5\pi}{8}\right)$



Note:

- As $z^3 = i$ is the same as $z^3 = 1$, except rotated about the origin $\frac{\pi}{6}\, radians$ the case of $z^4 = i$ is the same as $z^4 = 1$ except rotated about the origin $\frac{\pi}{8}\, radians$

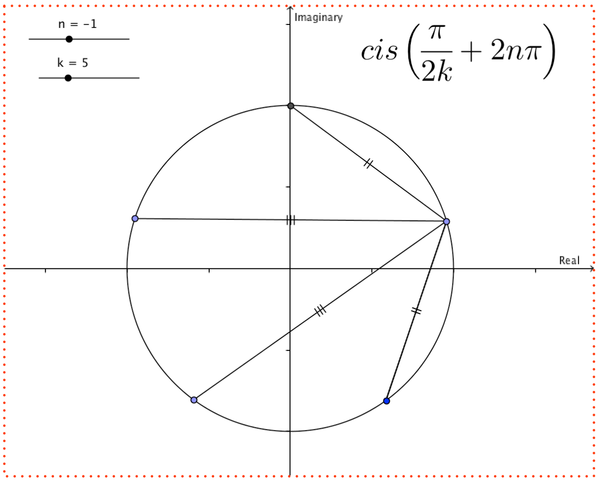
$z^5 = i$

$z^5 = cis\left(\frac{\pi}{2} + 2n\pi\right),\ n \in \mathbb{Z}$

$z = cis\left(\frac{\pi}{10} + \frac{2n\pi}{5}\right)$

$so,\ for\ n = -2,-1,0,1,2$

$z = cis\left(-\frac{7\pi}{10}\right), cis\left(-\frac{3\pi}{10}\right), cis\left(\frac{\pi}{10}\right), cis\left(\frac{5\pi}{10}\right), cis\left(\frac{9\pi}{10}\right)$



Note

- $z^5 = i$ is the same as $z^5 = 1$, except rotated about the origin $\frac{\pi}{10}$ *radians*

Once again, what about n of 2?

- $z^2 = i$ gives results of $z = cis\left(\frac{\pi}{4}\right), cis\left(-\frac{3\pi}{4}\right)$. As the difference between the arguments is π radians the roots are opposite ends of a diameter of the unit circle and so the distance is two which is equal to the value of n and so the general statement still holds.

Other Cases where the modulus is still one

- Here it was expected that the students did one or two explicit examples, although the explicit examples where not necessary if they could come up with the following generalizations

In solving $z^n = c$, where $|c| = 1$ and $\arg c = \theta$ then it follows that:

- we will still end up with an n-sided regular polygon in the unit circle
- the roots would be the same as in the case of $z^n = 1$, except that they would be rotated about the origin by an angle of $\frac{\theta}{n}$

Example:

- Find the roots for $z^3 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

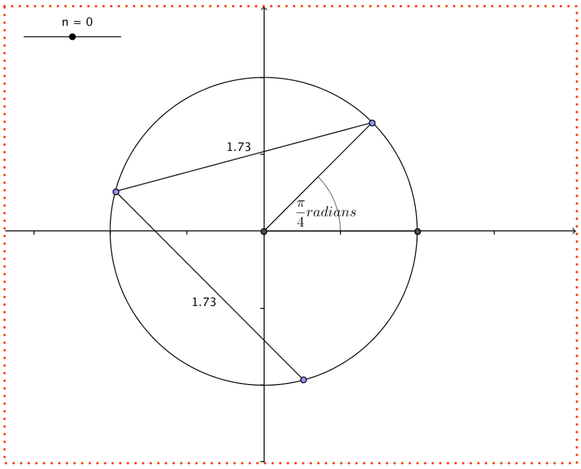
$z^3 = 1cis\left(\frac{3\pi}{4} + 2n\pi\right)$

$z = cis\left(\frac{\pi}{4} + \frac{2n\pi}{3}\right)$

$for\ n = -1,0,1$

$z = cis\left(-\frac{5\pi}{12}\right), cis\left(\frac{\pi}{4}\right), cis\left(\frac{11\pi}{12}\right)$

As the diagram to the right illustrates, the roots still form an equilateral triangle in the unit circle



and the roots are the same as in the case of $z^n = 1$ only rotated about the origin by an angle of $\frac{\pi}{4}$ radians, which comes from $\frac{3\pi}{4} \div 3$.

So, for all cases of $z^n = c$, where $n \in \mathbb{Z}, n \geq 2$ and $|c| = 1$ the general statement holds. The product of all the distances from one of the roots to the other $(n-1)$ roots will be n.

Modulus Other than One

- Here again, I think the students will have needed to do a few examples to begin to see a pattern. I would have expected two or three examples.
- The pattern to be discovered in this final bullet is as follows:
  - for $z^n = c$, where $n \in \mathbb{Z}^+, n \geq 3$ and $c \in \mathbb{C}$ then if the n roots are found and the distance from all of the roots to one specific root is found and then multiplied the product will be $n \times |c|^{\frac{n-1}{n}}$

General Proof of this Case

for $z^n = c$, let $\arg c = \theta$ and so if we solve by changing c into polar form and then applying De Moivre's theorem we will have the following:

$$z^n = |c| cis(\theta + 2k\pi), k \in \mathbb{Z}$$
$$z = \left[|c| cis(\theta + 2k\pi)\right]^{\frac{1}{n}}$$
$$z = \sqrt[n]{|c|} cis\left(\tfrac{\theta}{n} + \tfrac{2k\pi}{n}\right)$$
$$for\ k = 0,1,2,\ldots,n-1$$
$$z = \sqrt[n]{|c|} cis\left(\tfrac{\theta}{n}\right), \sqrt[n]{|c|} cis\left(\tfrac{\theta}{n} + \tfrac{2\pi}{n}\right), \ldots, \sqrt[n]{|c|} cis\left(\tfrac{\theta}{n} + \tfrac{2(n-1)\pi}{n}\right)$$

At this point it can be noted that the roots will be located symmetrically around the origin, but not on the unit circle, but on a circle with radius of $\sqrt[n]{|c|}$ units. The first root will also be at an angle of $\frac{\theta}{n}$, which follows from our previous result of cases where the modulus of c is one.

Now, in finding the distances from all the roots to the root $\sqrt[n]{|c|} cis\left(\frac{\theta}{n}\right)$. To illustrate what will happen, I will show the result from finding the distance from $\sqrt[n]{|c|} cis\left(\frac{\theta}{n}\right)$ to $\sqrt[n]{|c|} cis\left(\frac{\theta}{n} + \frac{2\pi}{n}\right)$:

$$d=\sqrt{\left[\sqrt[n]{|c|}\cos\left(\tfrac{\theta}{n}\right)-\sqrt[n]{|c|}\cos\left(\tfrac{\theta}{n}+\tfrac{2\pi}{n}\right)\right]^2+\left[\sqrt[n]{|c|}\sin\left(\tfrac{\theta}{n}\right)-\sqrt[n]{|c|}\sin\left(\tfrac{\theta}{n}+\tfrac{2\pi}{n}\right)\right]^2}$$
$$=\sqrt{\left[\sqrt[n]{|c|}\left(\cos\left(\tfrac{\theta}{n}\right)-\cos\left(\tfrac{\theta}{n}+\tfrac{2\pi}{n}\right)\right)\right]^2+\left[\sqrt[n]{|c|}\left(\sin\left(\tfrac{\theta}{n}\right)-\sin\left(\tfrac{\theta}{n}+\tfrac{2\pi}{n}\right)\right)\right]^2}$$
$$=\sqrt{|c|^{\frac{2}{n}}\left(\cos\left(\tfrac{\theta}{n}\right)-\cos\left(\tfrac{\theta}{n}+\tfrac{2\pi}{n}\right)\right)^2+|c|^{\frac{2}{n}}\left(\sin\left(\tfrac{\theta}{n}\right)-\sin\left(\tfrac{\theta}{n}+\tfrac{2\pi}{n}\right)\right)^2}$$
$$=\sqrt{|c|^{\frac{2}{n}}\left[\left(\cos\left(\tfrac{\theta}{n}\right)-\cos\left(\tfrac{\theta}{n}+\tfrac{2\pi}{n}\right)\right)^2+\left(\sin\left(\tfrac{\theta}{n}\right)-\sin\left(\tfrac{\theta}{n}+\tfrac{2\pi}{n}\right)\right)^2\right]}$$
$$=|c|^{\frac{1}{n}}\sqrt{\left(\cos\left(\tfrac{\theta}{n}\right)-\cos\left(\tfrac{\theta}{n}+\tfrac{2\pi}{n}\right)\right)^2+\left(\sin\left(\tfrac{\theta}{n}\right)-\sin\left(\tfrac{\theta}{n}+\tfrac{2\pi}{n}\right)\right)^2}$$

In the end we would have $n-1$ distances to multiply, and all of these distances would have $|c|^{\frac{1}{n}}$, which means that the final product will contain $|c|^{\frac{n-1}{n}}$.

Furthermore, from the cases when $|c|=1$ it was shown that the product of the distances would equal n. Hence,

$$\sqrt{\left(\cos\left(\tfrac{\theta}{n}\right)-\cos\left(\tfrac{\theta}{n}+\tfrac{2\pi}{n}\right)\right)^2+\left(\sin\left(\tfrac{\theta}{n}\right)-\sin\left(\tfrac{\theta}{n}+\tfrac{2\pi}{n}\right)\right)^2}\times\sqrt{\left(\cos\left(\tfrac{\theta}{n}\right)-\cos\left(\tfrac{\theta}{n}+\tfrac{4\pi}{n}\right)\right)^2+\left(\sin\left(\tfrac{\theta}{n}\right)-\sin\left(\tfrac{\theta}{n}+\tfrac{4\pi}{n}\right)\right)^2}\times...$$
$$...\times\sqrt{\left(\cos\left(\tfrac{\theta}{n}\right)-\cos\left(\tfrac{\theta}{n}+\tfrac{(2n-2)\pi}{n}\right)\right)^2+\left(\sin\left(\tfrac{\theta}{n}\right)-\sin\left(\tfrac{\theta}{n}+\tfrac{(2n-2)\pi}{n}\right)\right)^2}=n$$

And so, the product of all the distances will be $n\times|c|^{\frac{n-1}{n}}$