WORK-ENERGY PART II

A collection of facts about conservative forces and potentials, from several references.

Conservative force

Work done by a conservative force

- is independent of the path of the object acted on by the force;
- depends only on the start and end points of the motion of the object;
- will be zero if the start and end points are the same (closed path);
- is reversible;
- can be expressed as the difference between the initial and final values of a potential energy function.

If there is no change in the kinetic energy (KE) as a particle moves through a closed path then the force acting on it is conservative. Inversely, if there is a change in KE then at least one force acting must be nonconservative. A force is conservative if the work done by the force on a particle that moves between two points depends only on those points and not on the path followed. If the work depends on the path then the force acting is nonconservative.

The condition that the net work done by a force on an object over a closed path is zero defines a conservative force under *all* circumstances. A conservative force always acts to "push" a system toward lower potential energy.

A conservative force must do both positive and negative work along a closed path. By contrast, since the work done by friction is *always negative* (the friction force is always in direction opposite that of the motion), friction's work does not change sign and thus cannot sum to zero around a closed path. A nonconservative force, such as friction, cannot be represented by a potential energy function. A friction force depends on the direction of the object's velocity and thus is not conservative.

Any one-dimensional force that depends only on position is conservative. Any constant force is conservative. In one dimension, if the impressed force F is a function of position only, it is said to be conservative, and then the sum of the kinetic and potential energies is constant. Constraint forces act at right angles to the motion and so do no work and thus do not contribute to potential energy.

If a force is conservative then a potential (energy) function can be found for it.

Potential energy

"All phenomena depend on the variations of energy and not on its absolute value" ... Maxwell

The potential function was introduced as an auxiliary mathematical quantity, useful in finding a force. The concept of potential and the formal theories for dealing with it were developed for application to problems in gravitational attraction. Potentials are associated with conservative forces only. The work done by a conservative force is equal to the loss in potential energy of the object upon which the force acts.

Potential times a multiplicative constant is the potential energy (PE) of the system. The PE of a system represents a form of stored energy that can be fully recovered and "converted" into the energy of motion, kinetic energy. The zero level of PE is arbitrary, and only *changes* in PE have any physical meaning. . The PE may be regarded as the stored ability of a conservative force to do work in order to return an object from its current position back to the position of zero PE.

The work done by conservative *vector* forces can be re-expressed in terms of a simple *scalar* function of position called the potential. This can greatly simplify the analysis of 3D systems, since the scalar potential is easier to work with than the vector force. In 1D the advantage is not so clear.

This scalar potential is especially useful for systems with many forces or objects, such that the total potential is just a scalar sum, rather than a more complicated vector resultant. For example, the gravitational force "field" defined by several masses is more easily described by a potential sum, and then the force at a point is found from the gradient of this potential (as opposed to doing all the vector resultants for the several masses). Potentials are used to provide a convenient means for calculating the force on an object, but in some problems it may be easier to find the force directly, rather than by finding a potential and then taking its gradient (derivative, in 1D).

If the net work done by a force on an object is zero over any path that starts and ends at the same place, then a potential (energy) function can be defined for that force. That is, the only way to assign a unique value to the potential energy is if the work integral around a closed path vanishes. If this is the case then the work done along a path from A to B would be independent of the specific path taken, and then would equal the potential energy *lost*, which also equals the kinetic energy *gained*. Thus the total mechanical energy is constant.

Potential energy is a property of a *system*, not an object by itself. PE is energy associated with the configuration of a system in which a conservative force acts. In some cases "configuration" refers to the relative position of the objects in the system. The PE is found by calculating the work done in moving a system slowly, *without changing the kinetic energy,* from the zero point of the PE to the desired position. Doing positive work requires reducing the PE of the system (like taking money out of the bank).

The component of a conservative force in any direction is equal to the spatial rate of decrease in the potential in that direction. The potential energy is a function of position, whose negative derivative gives the force at that position. In more than one dimension, the scalar derivative of the potential is replaced by the vector gradient of the potential.

Basic mathematical relations

In one dimension, say, x, the change in potential energy U as an object moves from x_i to x_f under the influence of force F is, with θ the angle between the force and the displacement,

$$\Delta U = U_f - U_i = -\int_{x_i}^{x_f} F(x)\cos(\theta)\,dx \tag{1}$$

and we can define a potential energy function as

$$U(x) = U_0 - \int_{x_0}^{x} F(\alpha)\cos(\theta)\,d\alpha \tag{2}$$

where α is a dummy integration variable. To find the force from the potential energy function we use

$$F = -\frac{dU}{dx} \tag{3}$$

Mathematical details (more advanced, optional)

If a force is conservative, then it is the case that

$$\nabla \times \mathbf{F} \equiv \mathrm{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \mathbf{0} \tag{4}$$

This is a necessary and sufficient condition for a force **F** which is a function of position only to be conservative, and for the existence of the potential *V(x,y,z)*. It can also be shown, by Stokes's Theorem, that this condition means that the line integral (i.e., the work integral) from some start point A to some end point B is independent of the path between A and B. Further, under this condition, if A and B are the same point (a closed path) then the net work around the path from A back to A is zero. This condition also means that the vector force **F** can be found from the gradient of the scalar function V, that is

$$\mathbf{F} = -\nabla V = -\left(\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right) \tag{5}$$

and we can evaluate the work line integral in a simple way (here using the position vector **r** instead of Cartesian components):

$$\int_A^B \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = V(A) - V(B) \quad \text{if} \quad \mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r}) \tag{6}$$

since the result does not depend on the path between A and B. Note that the work done by **F** in moving the object from A to B is equal to the *decrease* in PE; a change in PE is the *negative* of the work done on the object. Eq(6) is called the Fundamental Theorem of Line Integrals, by analogy to the Fundamental Theorem of Calculus.

When a vector field, like a force defined at any point in some region of space, meets condition (4) and is thus path-independent, we can define the potential energy of the system consisting of the force and an object influenced by that force. When that object moves to another position, the potential energy changes by an amount equal to the work done by the force, and this work depends only on the starting and ending positions. If the work done was not path-independent, then the potential energy would depend on both the object's current position *and* on how it got there, making it impossible to define a useful potential energy.

Energy diagrams

We can graph the *U(x)* function from Eq(2), or it may be given separately in a problem. These graphs contain much useful information about the motion of an object in a conservative field. A separate handout discusses these diagrams, and gives several examples. There is also a computer simulation that we will use to explore these graphs. Here is some mathematics useful in the analysis of these diagrams.

Points of equilibrium

$$\frac{dU}{dx} = 0 \qquad \frac{d^2U}{dx^2} > 0 \qquad \text{stable equilibrium}$$

$$\frac{dU}{dx} = 0 \qquad \frac{d^2U}{dx^2} < 0 \qquad \text{unstable equilibrium}$$

$$\frac{dU}{dx} = 0 \qquad \frac{d^2U}{dx^2} = 0 \qquad \text{neutral equilibrium}$$

In the last case, the equilibrium will be stable if the first derivative found to be different from zero is of even order and positive, otherwise it is unstable. Only if all higher derivatives are zero is the equilibrium truly neutral. These cases correspond to a minimum, a maximum, and an inflection point in the graph of *U(x)*, respectively.

Turning points

These points are found by using

$$U(x) = E_{total}$$

and solving for the *x* position(s). Depending on the form of *U(x)* this could be a very complicated process, and using the TI calculator's "Intersection" capability can make this much easier. Of course the calculators can also be used to find the extrema, and the force at a point.