A FIRST ORDER MATHEMATICAL MODEL OF THE CRUST OF THE EARTH FOR CALCULATING STRESSES DUE TO TANGENTIAL FORCES

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Abstract. The geophysical problem which is the original reason for developing the present mathematical model is that of wide displacements of geographical poles. A theory forecasting such a phenomenon was formulated by Hapgood and Campbell in the 1950's and rests on the hypothesis of the existence of a force having a component tangent to the surface of the earth. The particular force they thought to be the responsible for this phenomenon was rather controversial; moreover no detailed calculation of the stresses in the crust was given.
Here a first order mathematical model is proposed for the crust of the earth subjected to a generic force having a component on the tangent plane. The two dimensional continuum is studied in the Riemannian surface of the ellipsoid. The present model could also be of some use in astrophysics when dealing with microstarquakes on the neutron stars.

Keywords: Geophysics ; crust of the earth; twodimensional continuum.

INTRODUCTION

In the 1950's Hapgood and Campbell (1956, 1958) faced an ancient problem of geophysics: the wide displacement of the geographical poles. They sustained that during the hystory of our planet displacements of thousands of kilometers should have taken place during intervals of time very short as compared with geological times (they suggested periods of movement of 2-3 thousand years every 40-50 thousand years). Such a statement was supported by climatic, paleontologic and geophysical arguments. The idea was not new; it was considered and discarded even by Maxwell (1857). New was however the mechanism they suggested.
Let us consider the earth like an ellipsoid of revolution which is rotating around its minor axis. Due to the well known phenomenon of the stiffness of the gyroscopical axis, very strong forces should be needed in order to have appreciable displacements of the axis of rotation. That was the reason why the idea of wide displacements of the poles was abandoned for a long time.
On the other hand Hapgood and Campbell considered movements of the crust as a whole. The crust of the earth is floating on the magma. A force applied to the crust tends to move it. With a perfectly spherical earth and in the absence of static friction the movement would begin. With an oblate ellipsoid, if the moment of the force with respect to the center has a null component along the rotational axis, the movement is not allowed and the crust is subjected to stresses. If they reach the ultimate strength the crust will break and the movement will begin. Hapgood and Campbell proposed a force as a candidate to be the responsible for this phenomenon. They observe that the wide expance of the antarctic ice has its center of mass which is not on the axis of rotation. If the ice is not isostasated, a centrifugal force will be originated in the reference system at rest with the earth. Such a force will have a component tangent to the meridian. The theory was praised by Einstein himself but was later abandoned mainly for two reasons. One of them is the controversial hypothesis that the antarctic polar ice is not isostasated. The other was the lack of a mathematical model able to lead to detailed calculations of the stresses in the crust.
Here a mathematical model is studied for the crust of the earth subjected to forces applied to a small part of it (independently of the hypotheses on the nature of these forces).

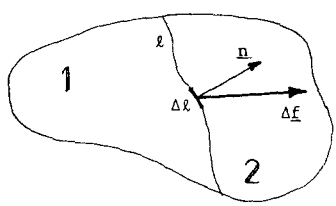
THE MODEL

In order to give a mathematical model of the physical system let us introduce strong simplifying hypotheses which will be progressively weakened in future developments.
-The shape of the earth is considered like an ellipsoid of revolution where B is the minor semi-axis while A is the radius of the equatorial circle.
-The crust of the earth is regarded like a shell, that is like a continuum, perfectly flessible system which can be geometrically represented by a two dimensional surface.
-The properties of the shell are assumed to be isotropic and homogeneous. Obviously this hypothesis is not verified for the crust of the earth and will therefore be the first one to be abandoned in a second order model.
-Let the closed shell be acted upon by a system of forces applied to a part of the surface small if compared to the whole surface.
-Let the resultant of this system of forces have no component on the tangent to the local parallel; it will only have a

component along the normal to the surface and a component along the tangent to the meridian.
-Let us moreover suppose that the contact between the ellipsoid and the shell is frictionless. This hypothesis is surely not true during the movement but it seems plausible for the static calculations of the stresses. The constraint force will therefore be normal to the surface in each point.
When all such hypotheses are introduced we will say that the first order model is considered.
The problem could be treated as a threedimensional one in the Euclidean space. Here it appears more convenient to treat it as a <u>twodimensional problem</u> in the Riemannian geometry of the twodimensional surface.

THE CLASSICAL PROCEDURE

Let us shortly outline the deduction of the general formulas just to show where there are differences with respect to the usual plane case.
Let σ be a piece of the shell , $d\sigma$ the surface element and $\underline{F}d\sigma$ the external force acting on it. The stress in a point P is defined by cutting the shell with a regular line ℓ passing through P and dividing the surface in two regions 1 and 2. Let $\underline{n}$ be the normal to the line ℓ in P going from 1 to 2.



If $\Delta\ell$ is a finite element of ℓ containing P and $\underline{f}$ is the force that the region 2 exerts on the region 1 through $\Delta\ell$, we define the vector

(1) $$\underline{p}_{(n)} = \lim_{\Delta\ell \to 0} \Delta\underline{f}/\Delta\ell \quad .$$

Here too we have the Cauchy's formula

(2) $$\underline{p}_{(n)} = \underline{p}_i n^i \quad ,$$

where the summation symbol Σ is suppressed adopting the Einstein convention and $\underline{p}$ is the vector associated to an element normal to the axis x^i .
Notice that one can use a generic coordinate system (x^i) ; if the surface is not developable, like in our case, it is not even possible to choose a Cartesian reference frame.
The indefinite equations of equilibrium can be obtained by the standard procedure, that is by requiring that the resultant and the moment of the forces acting on a generic element of the surface are null. One gets

(3) $$\underline{F} + \underline{p}_i{}^{;i} = 0 \quad ,$$

(4) $$\underline{P}^{;i} \times \underline{p}_i = 0 \quad ,$$

where semicolons stand for covariant derivatives.

If $\underline{N}$ is the unit vector normal to the surface, one easily obtains that

(5) $$\underline{N} \cdot \underline{p}_i = 0 \quad ,$$

that is the stresses are tangent to the shell and are therefore completely determined by the components of the <u>stress tensor</u>

(6) $$p_{ij} = \underline{p}_i \cdot \underline{P}_{;j} \quad . \qquad (i,j=1,2)$$

One gets moreover the symmetry of such a tensor i.e.

(7) $$p_{ij} = p_{ji} \quad .$$

By setting

(8) $$F_j = \underline{F} \cdot \underline{P}_{;j} \quad ,$$

(9) $$\Phi = \underline{F} \cdot \underline{N} \quad ,$$

equations (3) become

(10) $$F_j + p_{ij}{}^{;i} = 0 \quad ,$$

(11) $$\Phi + \underline{p}_i{}^{;i} \cdot \underline{N} = 0 \quad .$$

This is true in general. Now we have to take into account that in our hypotheses the shell leans on a rigid and given surface. Therefore the metric of the surface is given.
The F_i can be regarded as given. The unknowns of our problem are the three components of the symmetric stress tensor p_{ij} and Φ which depends on the constraint force.
In order to determine p_{ij} we have the two indefinite eqs.(10) only. The eq.(11) can be used in order to obtain Φ once p is known.
The situation is analogous (but not equal) to that of the usual equilibrium of a continuum. In the former case one has a threedimensional body in the threedimensional Euclidean space, while the intrinsic study of the equilibrium of a shell leaning on a given surface leads to a twodimensional problem in a Riemannian twodimensional surface (that is in general, not plane nor developable).
By tensor calculus the two problems are treated in a very similar way. There are however some important differences. For example the derivatives with respect to two different coordinates do not commute. In other words, if A_i are the components of a vector,

$$A_{i;jk} \neq A_{i;kj} \quad .$$

We notice that eqs.(10) are not sufficient to determine the stress tensor p_{ij}. One has to take into account the nature of the shell and its deformations. For a regular and infinitesimal displacement whose components are s_i the deformation tensor is given by

(12) $$\xi_{ik} = \tfrac{1}{2}(s_{i;k} + s_{k;i}) \quad .$$

For an ordinary <u>elastic</u> body, referring both the deformation tensor and the stress tensor to the not deformed configuration, we have the Hook's low

(13) $$p_{ik} = c_{ikrs}\,\xi^{rs} \quad .$$

The symmetry of both tensors p_{ik} and ξ^{rs} requires the symmetry of the tensor

c_{ikrs} with respect to the two couples of indexes:

(14) $$c_{ikrs} = c_{kirs} \quad ,$$

(15) $$c_{ikrs} = c_{iksr} \quad .$$

We made the hypothesis of dealing with an <u>isotropic</u> body. This implies

(16) $$c_{ikrs} = Aa_{ik}a_{rs} + Ba_{ir}a_{ks} + Ca_{is}a_{kr} \; ,$$

where A,B,C are arbitrary scalar functions.
If moreover the body is <u>homogeneous</u> it is

(17) $$c_{ikrs} = \lambda a_{ik}a_{rs} + \mu(a_{ir}a_{ks} + a_{kr}a_{is}) ,$$

where λ and μ are the Lame's constants (which are related to the elastic modulus E and to the Poisson's ratio m by the well known formulas).
Notice that in the overmentioned formulas the indexes assume the values 1 and 2 only and a_{ik} are the components of the fundamental metric tensor of the Riemannian surface.
One obtains the stress tensor

(18) $$p_{ik} = \lambda\theta a_{ik} + 2\mu_{\ ik} \quad ,$$

where $\theta = a^{rs}\xi_{rs}$.
The equations of equilibrium are therefore given by

(19) $$F_j + \lambda\theta_{;j} + \mu(s_{j;k}^{\ \ .k} + s_{k;j}^{\ \ .k}) = 0.$$

If the surface is not developable it is no more true that

(20) $$s_k^{\ ;k}{}_j = s_{k;j}^{\ \ .k} \quad ,$$

and one does not obtain the usual formulas as in the plane case. Here, if K is the Gaussian curvature of the surface, it is

(21) $$s_{k;j}^{\ \ k} = s_k^{\ ;k}{}_j - K\varepsilon_k^{\ r}\varepsilon^k_{\ j}s_r \quad ,$$

where $\varepsilon^r_{\ k}$ is the completely skewsymmetric tensor.
The equations of equilibrium become

(22) $$F_j + (\lambda+\mu)s_k^{\ ;k}{}_j + \mu s_{j;k}^{\ \ k} + \mu K s_j = 0 \quad .$$

By integrating eqs.(22) (with the boundary conditions) the displacements s_i will be found which, put in eqs.(18), give the stress tensor.
In our model we suppose that the force F_j is present in a small region only. It can therefore be represented by a δ distribution. Otherwise one can, more physically, cut a little hole corresponding to the region where F_j is present and regard the edge of the hole as the boundary of the problem that in this way becomes homogeneous. On this edge a force per unit length will be applied such that the resultant is equal to the applied total force.
The mathematical difficulties in integrating such equations in our particular case are however intimidating.
Moreover, as we have seen, eqs.(22) hold only when the shell is elastic homogeneous and isotropic. When dropping these hypotheses all the previous results should be abandoned and the calculations should be made in a different way from the very beginning. That is why a different approach to the problem has been here considered.

A DIFFERENT PROCEDURE

Let us return to the general equations of equilibrium before introducing in it the stress-strain relationship, that is to

(23) $$F_j = p_{ij}^{\ \ ;j} \quad .$$

Let us consider the region where $F_j = 0$. We have

(24) $$p_{ij}^{\ \ ;j} = 0 \quad .$$

The different procedure consists in integrating eqs.(24) first. In other words we want the most general divergenceless tensor in the Riemannian twodimensional surface of the ellipsoid. To this purpose some old results by Bianchi (1922), Finzi (1934) and Graiff (1959) will be used.
Let us remember that for a surface with total curvature K = const one can find the most general divergenceless tensor

(25) $$\Gamma_{ik} = a_{ik}(h_{;s}^{\ \ s} + Kh) - h_{;ik} \quad ,$$

where h is an arbitrary function of the place.
On the other hand, if K is not constant, like in our case, it is

(26) $$\Gamma_{ik}^{\ \ ;k} = hK_{;i} \quad .$$

One has therefore to search for the most general tensor D_{ik} whose divergence is

(27) $$D_{ik}^{\ \ ;k} = -hK_{;i} \quad .$$

The solution will be

(28) $$p_{ik} = \Gamma_{ik} + D_{ik} \quad .$$

The problem of finding D_{ik} is particularly simple in the case of the present model since the line K = const are geodetically parallel. We remember that a system of ∞^1 lines is called <u>geodetically parallel</u> if their orthogonal trajectories are geodesics of the surface. In the twodimensional surface of an ellipsoid the "parallels" give a family of geodetically parallel lines since the meridians are their orthogonal tajectories and they are geodesics of the surface.
By setting

(30) $$H = K_{;i}K^{;i} \quad ,$$

it can be shown (Bianchi, 1922) that if

(31) $$\varepsilon^{ij}K_{;i}H_{;j} = 0 \quad ,$$

and K is not a constant, then

(32) $$H = \Omega(K) \quad ,$$

and the lines K = const are geodetically parallel.
The overmentioned conditions are verified in our case. Let us therefore consider the symmetric tensor

(33) $$D_{is} = gK_{;i}K_{;s} \quad ,$$

where g is an arbitrary scalar. It is

(34) $$D_{is}^{\ \ ;s} = (g^{;s}K_{;s} + gK_{;s}^{\ \ s})K_{;i} + \tfrac{1}{2}gH_{;i} \quad .$$

Taking into account eq.(32) it is

$$(35)\qquad D_{is}{}^{;s} = \{ g^{;s} K_{;s} + gK_{;s}{}^{s} + \tfrac{1}{2}g\Omega'(K) \} K_{;i} \quad .$$

By comparing eqs.(35) with eqs.(27) one can set

$$(36)\qquad -h = g^{;s} K_{;s} + gK_{;s}{}^{s} + \tfrac{1}{2}g\Omega'(K) \ ,$$

and we finally get

$$(37)\qquad p_{ik} = gK_{;i} K_{;k} + a_{ik}(h_{;r}{}^{r} + Kh) - h_{;ik} \ ,$$

which is divergenceless.
It can be shown that eqs.(37) give the most general divergenceless tensor on the surface.
It can be moreover noticed that this expression contains an arbitrary scalar function g together with its derivatives up to the third order one.
Let us now come to the particular geometry of our case, that is to the ellipsoid.
First of all a system of intrinsic coordinates has to be chosen. It is convenient to chose two angles

$$(38)\qquad x^{1} = \phi \ ,$$

$$(39)\qquad x^{2} = \theta \ ,$$

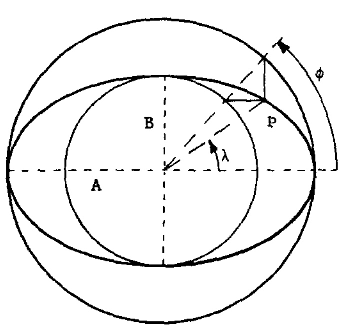
such that in a cartesian threedimensional frame y^i the equations of the surface of the ellipsoid is given by

$$(40)\qquad y^{1} = A \cos\phi \cos\phi \ ,$$

$$(41)\qquad y^{2} = A \cos\phi \sin\phi \ ,$$

$$(42)\qquad y^{3} = B \sin\phi \ .$$

The angle θ gives the longitude of the generic point P and is the angle formed by the plane β passing through the point P and the axis of symmetry with the plane containing the reference meridian.
In the overmentioned plane β the position of the point P is obtained by the following construction



The angle ϕ does not therefore coincide with the latitude λ of the point P. The latter is however related to ϕ by

$$(44)\qquad \mathrm{tg}\,\phi = A/B\ \mathrm{tg}\,\lambda \ .$$

The covariant components of the fundamental tensor are given by

$$(45)\qquad a_{ik} = (\partial P/\partial x^{i})(\partial P/\partial x^{j}) = (\partial y_{\alpha}/\partial x^{i})(\partial y^{\alpha}/\partial x^{j}).$$

One gets

$$(46)\qquad a_{11} = A^{2}\sin^{2}\phi + B^{2}\cos^{2}\phi \ ,$$

$$(47)\qquad a_{12} = a_{21} = 0 \ ,$$

$$(48)\qquad a_{22} = A^{2}\cos^{2}\phi \ ,$$

and the contravariant components

$$(49)\qquad a^{11} = 1/(A^{2}\sin^{2}\phi + B^{2}\cos^{2}\phi),$$

$$(50)\qquad a^{12} = a^{21} = 0 \ ,$$

$$(51)\qquad a^{22} = 1/(A^{2}\cos^{2}\phi).$$

The Christoffel symbols are obtained by

$$(52)\qquad \{ij,k\} = \tfrac{1}{2}(\partial a_{ik}/\partial x^{i} + \partial a_{jk}/\partial x^{i} + \partial a_{ij}/\partial x^{k}) \ ,$$

and are

$$(53)\qquad \{11,1\} = (A^{2} - B^{2})\cos\phi \sin\phi \ ,$$

$$(54)\qquad \{12,2\} = - A^{2} \cos\phi \sin\phi \ ,$$

$$(55)\qquad \{22,1\} = A^{2} \cos\phi \sin\phi \ ,$$

$$(56)\qquad \{11,2\} = \{12,1\} = \{22,2\} = 0 \ .$$

The Riemann tensor can also be obtain by the formula

$$(57)\qquad R_{ilkr} = \partial\{ik,l\}/\partial x^{r} - \partial\{ir,l\}/\partial x^{k} - a^{hj}(\{ik,h\}\{lr,j\} - \{ir,h\}\{lk,j\}) \ .$$

We are interested only in the Gaussian curvature which is given by

$$(58)\qquad K = R_{1212} / D$$

where D is the determinant of the fundamental metric tensor. One obtains

$$(59)\qquad K = B^{2}/(A^{2}\sin^{2}\phi + B^{2}\cos^{2}\phi)^{2} \ .$$

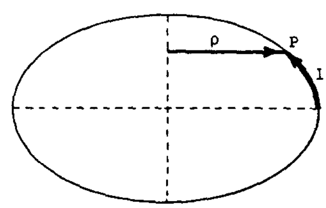
The same result can also be obtained by remembering that for a surface of revolution, if it is possible to put the metric in the form

$$(60)\qquad ds^{2} = dl^{2} + \rho^{2}(l)\, d\theta^{2} \ ,$$

then the curvature has the simple expression

$$(61)\qquad K = - d^{2}\rho/ \rho\, dl^{2} \ ,$$

In our case one obtains the form (60) by chosing the coordinates ρ and l as in the following figure (drawn in the plane β)



By these coordinates and eq.(61) one easily obtains eq.(59).
Now H can be obtained. It is

$$(62)\quad H = a^{is} K_{;i} K_{;s} = = 16B^4\{(A^2 - B^2)\sin\phi\cos\phi\}^2/(A^2\sin^2\phi + + B^2\cos^2\phi)^7 .$$

Moreover it is

$$(63)\quad \Omega'(K) = dH/dK = = 8B^2\{6(A^2 - B^2)^2\sin^2\phi\cos^2\phi - - (A^2 - B^2)(B^2\cos^4\phi - A^2\sin^4\phi)\}/ /(A^2\sin^2\phi + B^2\cos^2\phi)^5 .$$

By putting eqs.(59), (62) and (63) into eqs.(36) and (37) one gets the most general divergenceless tensor on the surface of the ellipsoid

$$(64)\quad p_{ik} = -h_{;ik} + + 16\,\delta^1_i\delta^1_k g B^4 (B^2 - A^2)^2\cos^2\phi\,\sin^2\phi\,/ /\,(A^2\sin^2\phi + B^2\cos^2\phi)^8 + + a_{ik}\{h_{;s}{}^s + hB^2/(A^2\sin^2\phi + B^2\cos^2\phi)^2\}.$$

where

$$(65)\quad -h = 4(\partial g/\partial\phi)(B^2 - A^2)\cos\phi\sin\phi\,/ /\,(A^2\sin^2\phi + B^2\cos^2\phi)^4 + + 4gB^2\{12(A^2 - B^2)^2\sin^2\phi\cos^2\phi + + 2(B^2 - A^2)(B^2\cos^4\phi - A^2\sin^4\phi) + + (A^2 - B^2)(A^2\sin^4\phi + B^2\sin^2\phi\cos^2\phi)\}\,/ /\,(A^2\sin^2\phi + B^2\cos^2\phi)^5 .$$

Notice that in calculating eq.(65) it has been taken into account that

$$(66)\quad K_{,i} = K_{;i} ,$$

where the comma followed by an index i stands for the partial derivative with respect to x^i while a semicolon stands for a covariant derivative, and that

$$(67)\quad K_{;i}{}^{i} = K_{,ij}a^{ij} - K_{,s}a^{sr}a^{ij}\{ij,r\}.$$

In the same way, when substituting eq.(65) into eqs.(64) in order to make them explicit, one has to take into account that

$$(68)\quad h_{;ij} = h_{,ij} - h_{,s}a^{rs}\{ij,r\}.$$

CONCLUDING REMARKS

A mathematical model for calculating the stesses in a shell leaning on a given ellipsoid and acted upon by a force applied to a small part of the shell has been given. A procedure different from the traditional one has been followed. The solution of the problem has been splitted in two steps the first of which consists in searching the general solution of the indefinite equations of equilibrium.
Equations (64) give the most general stress tensor on the surface of the given ellipsoid. An arbitrary scalar function g(,) together with its derivatives up to the third order is present; such a function has to be calculated by taking into account the properties of the material the shell is made out of, and the boundary conditions (the distribution of forces on the mentioned edge of the hole). This will be called the second step of the calculation of the stresses.
By comparing this approach to the one which is traditional (at least in the plane case) we can see that by splitting the problem in two steps it has been possible to solve the first one in an exact way. Moreover the results of the first step are independent of the kind of material the crust is made out of. This feature is particularly suitable for our case. Indeed when considering the set of the hypotheses of our model one realizes that the less plausible ones are just those of the isotropy and homogeneity of the material. These hypotheses will be retained for the first order model only and then will be dropped in order to introduce the experimental coefficients which are not isotropic nor homogeneous.
The advantage of the present procedure is that the results of the first step calculations will still be valid.
Finally one can observe that the model presented is not strictly linked to the problem of calculating stresses for the crust of the earth. It can be used, for example, in astrophysics in order to explain microstarquakes on the neutron stars. Indeed the current explanation (Baym and Pines, 1971; Pines and Shaham, 1972) is that they are caused by a misorientation between the axis of figure and the axis of rotation.

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