PRACTICE EXERCISES

1. Give a rigorous definition of a real irrational exponent of an integral base, and of any real base other than 1 or 0. (See Dresden's *An Invitation to Mathematics*, Chapter VI.)
2. Using Euler's formula, find the value of $e^{i\theta}$ when θ has the values $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, 2\pi$. What other values of θ give the same answers as those obtained for these specific values?
3. Show, under the definition of e^z, that $e^z \cdot e^{-z} = 1$.
4. Find the value of $10^{i\theta}$. Hint: $10 = e^{2.3026}$.
5. Write e^{10+7i} as a complex number.
6. Find the logarithms of the following complex numbers: $1, 5, -4, i, -i, \sqrt{3}+i, -\sqrt{2}-\sqrt{2}i, a+bi, ci$.
7. If α and β are complex numbers with modulus unity, show that all values of $\log_\beta \alpha$ are real numbers.
8. Evaluate x if $x = (2+2i)^i$.
9. Evaluate x if $i = x^{\sqrt{2}+\sqrt{2}i}$.
10. Evaluate x if $(\sqrt{3}+i) = i^x$.
11. Determine the value of each of the following: 1^1; i^i; $-i^i$; i^{-i}.
12. Solve the equation $x^3 = 1 + \sqrt{3}i$.
13. Solve the equation $x^3 = i$.
14. Solve the equations $i^3 = x$; $i^x = 3$.
15. Solve $1^x = 2$ for all possible values of x.
16. Devise some possible way of giving graphical representation to the function e^{ix} for only real values of x.

Complex roots of the quadratic equation

Consider the quadratic equation:

$$x^2 + 3x + 4 = 0$$

The roots of this equation are found to be the complex numbers:

$$\tfrac{1}{2}(-3 \pm \sqrt{-7})$$

If we draw the usual graph of the function

$$y = x^2 + 3x + 4,$$

we obtain the parabola shown in Figure 194. Since this graph does not intersect the x-axis, we say that there is no real value of x for which $y = 0$. A high-school student, however, very frequently gets the idea that the graph indicates that the equation has no roots, and hence he uses the word imaginary in the sense that the roots do not exist. We shall give a method of graphical representation that will remove this misunderstanding of complex roots.

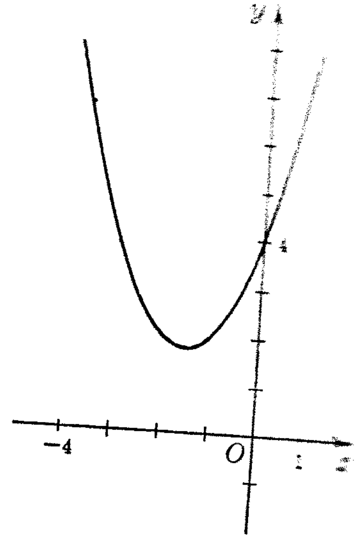

FIG. 194

Let the function $y = x^2 + 3x + 4$ have only real values of y; x may be real or complex. To tabulate a set of values, solve the equation for x in terms of y. Thus:

$$x = \frac{-3 \pm \sqrt{4y - 7}}{2} = -\frac{3}{2} \pm \frac{\sqrt{4y - 7}}{2}$$

For each real value given y, there are two values for x which are symmetrical to the line $x = -\frac{3}{2}$. For values of $y < \frac{7}{4}$ the values of x are still symmetrical to $x = -\frac{3}{2}$, but no longer real values, and hence cannot be located in the real plane.

x	[illegible]
$0, -3$	[illegible]
$-\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$	[illegible]
$-\frac{3}{2} \pm \frac{1}{2}$	[illegible]
$-\frac{3}{2} \pm 0$	[illegible]
$-\frac{3}{2} \pm \frac{1}{2}\sqrt{3}i$	[illegible]
$-\frac{3}{2} \pm \frac{1}{2}\sqrt{7}i$	[illegible]
$-\frac{3}{2} \pm \frac{1}{2}\sqrt{11}i$	− [illegible]
$-\frac{3}{2} \pm \frac{1}{2}\sqrt{15}i$	− [illegible]

To graph all the values of x, we consider a horizontal plane for complex values of x (the Argand diagram) and a real axis y perpendicular to the x-plane at the origin. For values of $y \geqq \frac{7}{4}$ we obtain the real branch of the quadratic, the curve CVD shown in Figure 195. For values of $y < \frac{7}{4}$ the values of x lie in the plane $x = -\frac{3}{2}$ and we obtain the normal branch of the quadratic, the curve EVF. The planes of these curves are perpendicular to each other. The curve crosses the x-plane at two points R_1 and R_2 which have the values $\frac{-3 \pm \sqrt{7}i}{2}$ and which represent the roots of the equation $x^2 + 3x + 4 = 0$.

Now consider the problem more generally. What are the conditions that complex values of x will satisfy

$$y = x^2 + 3x + 4$$

where y is real? Let $x = a + bi$ and substitute, obtaining:

$$y = (a + bi)^2 + 3(a + bi) + 4$$

$$y = a^2 - b^2 + 3a + 4 + i(2ab + 3b) \tag{1}$$

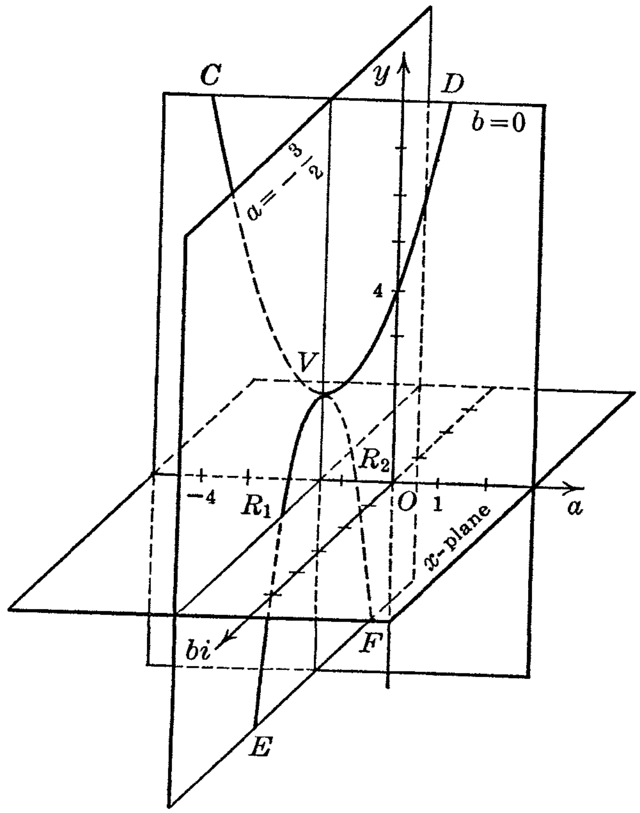

FIG. 195

Since y is real, the coefficient of i in the right-hand member is zero, and we have $2ab + 3b = 0$, from which:

$$b = 0 \quad \text{and/or} \quad a = -\tfrac{3}{2}$$

In (1) we place $b = 0$ and obtain:

$$y = a^2 + 3a + 4$$

This is the real branch of the curve, CVD, in the plane $b = 0$. In (1) we place $a = -\frac{3}{2}$ and obtain:

$$y = -b^2 + \tfrac{7}{4}$$

This is the complex branch of the curve, EVF, lying in the plane $a = -\frac{3}{2}$. (Note that a is the real axis and b is the normal axis of the complex variable $x = a + bi$.) This lower branch intersects the x-plane in the two complex roots of the equation $f(x) = 0$.

Complex roots of the cubic equation

We now extend the same procedure to the graph of a cubic function. The term containing x^2 can always be removed from the cubic function by a proper translation of the axes, and so we consider a cubic of the form:

$$y = x^3 - x + 6$$

By substitution, it is easy to obtain one root of $y = 0$, as $x = -2$. The real graph of the function is shown in Figure 196, where the curve crosses the x-axis at only one point. Since a cubic equation has three roots, it is evident that the other two must be complex roots.

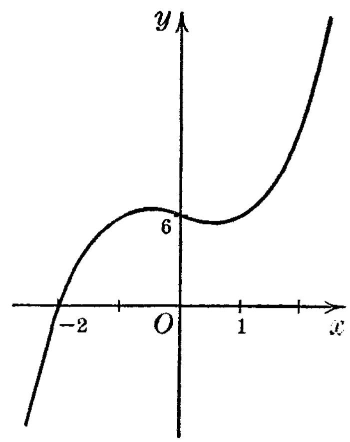

FIG. 196

To draw the complex branch for real values of y, we shall use the complex x-plane and real y-axis as before. If $x = a + bi$, the function yields:

$$y = (a + bi)^3 - (a + bi) + 6$$
$$y = a^3 - 3ab^2 - a + 6 + i(3a^2b - b^3 - b) \quad (1)$$

If y is real, the coefficient of i in the right-hand member is 0 and we obtain:

$$b = 0 \quad \text{and/or} \quad 3a^2 - b^2 - 1 = 0$$

If we place $b = 0$ in (1), we obtain

$$y = a^3 - a + 6$$

which is the real branch of the curve drawn in Figure 196. This figure is in the plane determined by the real x-axis and the y-axis.

If we place $3a^2 - b^2 - 1 = 0$, we obtain:

$$y = a^3 - 3a(3a^2 - 1) - a + 6$$
$$y = -8a^3 + 2a + 6$$

These values of y are plotted against the corresponding values of $x = a + bi$ in which a and b satisfy $3a^2 - b^2 - 1 = 0$. The last equation is that of a hyperbola in the x-plane. Since the values of y are plotted above (or below) the points on this hyperbola, the complex branch will be a space curve lying on the right hyperbolic cylinder. Figure 197

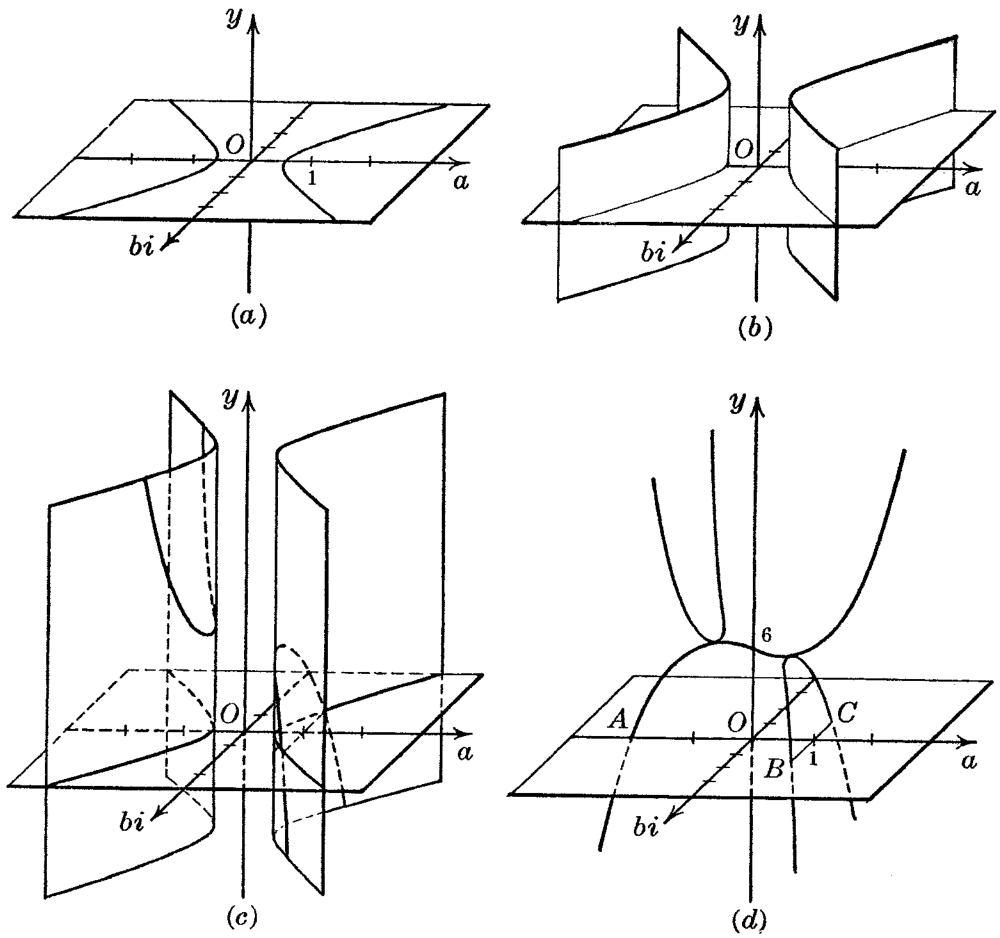

FIG. 197

shows respectively (*a*) the hyperbola in the x-plane; (*b*) the hyperbolic cylinder; (*c*) the complex branch of the cubic; and (*d*) the complete graph for real values of y.

Note that the complete graph intersects the $y = 0$ plane, or the x-plane, in three distinct points, namely $x = -2$ and $x = 1 \pm \sqrt{2}i$. These points are indicated by the letters A, B, and C respectively. From the total graph, it is evident that for any real value of y, a plane parallel to the x-plane will intersect the curve in three and always three points. All roots of $f(x) = y$ for real values of y are thus represented.

$$3a^2 - b^2 - 1 = 0$$
$$y = -8a^3 + 2a + 6$$

a	b	y
-3	± 5.1	216
-2	± 3.3	66
-1	± 1.4	12
$-\frac{1}{\sqrt{3}}$	0	$6 + \frac{2\sqrt{3}}{9} = 6.4$
$\frac{1}{\sqrt{3}}$	0	$6 - \frac{2\sqrt{3}}{9} = 5.6$
1	± 1.4	0
2	± 3.3	-54
3	± 5.1	-204

Further illustrations of complex roots

The same process can be extended to equations of the fourth and higher degree equations. For example, the function $y = x^4$ has for complex values of x the form:

$$y = (a + bi)^4 = a^4 - 6a^2b^2 + b^4 + i(4a^3b - 4ab^3)$$

If y is real, then $4ab(a^2 - b^2) = 0$ and $a = 0$, $b = 0$, and/or $a^2 - b^2 = 0$. If $b = 0$, $y = a^4$, which is the real branch of this quartic function. If $a = 0$, $y = b^4$, which is the normal branch of this quartic function.

If $a^2 = b^2$, $y = -4a^4$, and we have a pair of complex branches lying in the two perpendicular planes that bisect the quadrants of the x-plane. The complete graph is shown in Figure 198. Note that if y is positive and $x^4 = c$, we obtain a pair of real roots and a pair of conjugate normal roots. If y is negative and $x^4 = -c$, we obtain four complex roots.

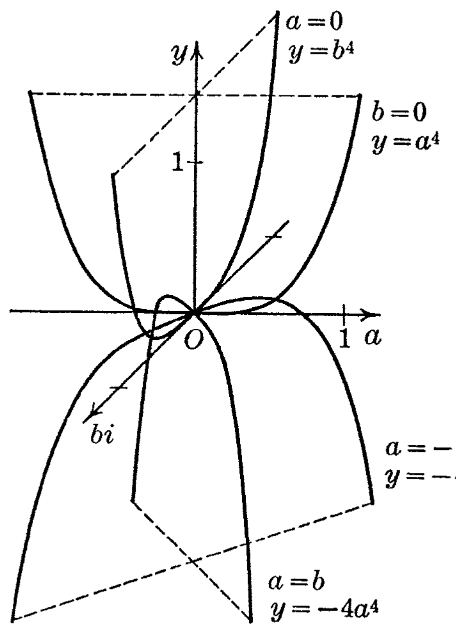

FIG. 198

As a further application consider the graph of $x^2 + y^2 = 25$, y real. Solving for x, we have $x = \pm\sqrt{25 - y^2}$. Substituting real values for y, we obtain a table of corresponding values for x, real if $-5 \leqq y \leqq +5$, otherwise normal values. These values can be plotted to give the complete curve. We can also seek the conditions imposed upon x to make y real. Letting $x = a + bi$, we obtain:

$$(a + bi)^2 + y^2 = 25$$

or

$$y^2 = 25 - a^2 + b^2 - 2abi$$

For y to be real, it is necessary and sufficient that y^2 be real and positive. If y^2 is real, the coefficient of i is zero and hence:

$$a = 0 \quad \text{or} \quad b = 0$$

If $b = 0$, $y^2 = 25 - a^2$ and we obtain the real circle.

If $a = 0$, $y^2 = 25 + b^2$ and we obtain the hyperbola in the normal plane. Hence, considered as a function of x, the complex branch of the

x	y
± 5	0
± 4.9	± 1
± 4.5	± 2
± 4	± 3
± 3	± 4
0	± 5
$\pm \sqrt{11}i$	± 6
$\pm \sqrt{17}i$	± 7

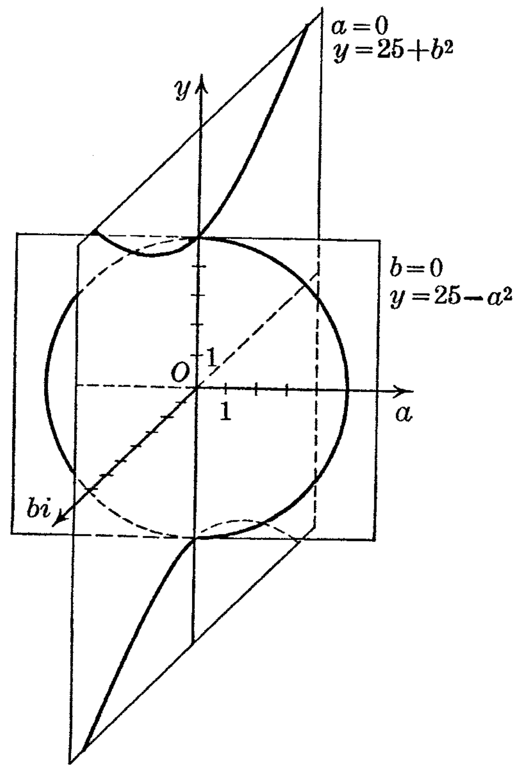

FIG. 199

circle, y real, is a hyperbola, and we can show that the interchange of the words circle and hyperbola gives a true statement also.

Discussion. At what place, if any, can the foregoing graphical presentation of complex roots be introduced into high school algebra?

PRACTICE EXERCISES

1. Draw the graph of $y = x^2 - 4x + 8$.
2. Find the branches of $y = ax^2 + bx + c$ for which y is real and x is a complex number.
3. Show that by proper translation and reflection the real and complex branches of the quadratic can be made identical.
4. Draw the graph of $y = x^3 - 15x + 4$. Find the value of y for those complex values of x for which y is real.

5. Draw the graph of $x^2 - y^2 = 1$, showing the real and the normal branches.
6. Solve $x^2 + y^2 = 25$ and $y^2 = \frac{16}{3}x$ as a set of equations.
7. Draw the graphs of the equations of Ex. 6 showing the real and the complex branches of both curves on the same set of coordinate axes. On this graph indicate the complex solutions to the equations.
8. Why is it impossible to graph the complex branch of a straight line for real values of y?
9. Draw the graph of $y = x^4 - 1$ showing all real and complex branches for real values of y.
10. Draw the graph of $y = x^4 - 2x^2 + 2$ showing all branches for real values of y.
11. Show the complex intersection of the circles $x^2 + y^2 = 9$ and $x^2 - 10x + 25 + y^2 = 1$.
12. Draw the complete graph of $y = x^5$. On it illustrate the roots of $x^5 = 32$ and $x^5 = -32$.