PHANTOM GRAPHS.

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Abstract. While teaching “solutions of quadratics” and emphasising the idea that, in general, the solutions of $ax^2 + bx + c = 0$ are obviously where the graph of $y = ax^2 + bx + c$ crosses the x axis, I started to be troubled by the special case of parabolas that do not even cross the x axis. We say these equations have “complex solutions” but **physically, where are these solutions**?
With a little bit of lateral thinking, I realised that **we can physically find the actual positions of the complex solutions of any polynomial equation** and indeed many other common functions! The theory also shows clearly and pictorially, why the complex solutions of polynomial equations with real coefficients occur in conjugate pairs.

Fig 1: The big breakthrough is to change from an ***x* AXIS…..**

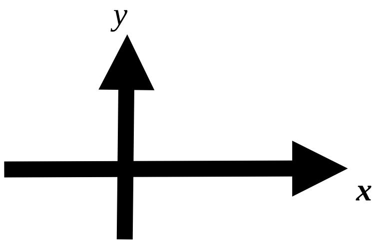
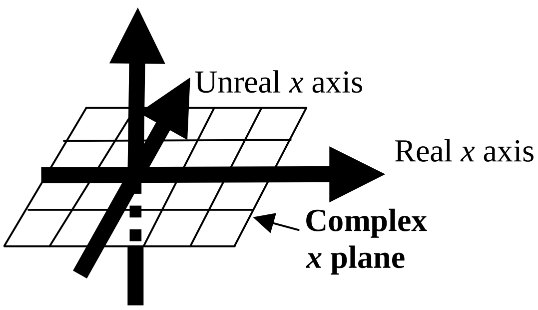


Fig 2

……..to a **complex *x* plane!**



Introduction. Consider the graph $y = x^2$.

We normally just find the positive y values such as: (±1, 1) , (±2, 4), (±3, 9) but we can also find **negative y values** even though the graph does not seem to exist under the x axis:

If $y = -1$ then $x^2 = -1$ and $x = \pm i$.

If $y = -4$ then $x^2 = -4$ and $x = \pm 2i$.

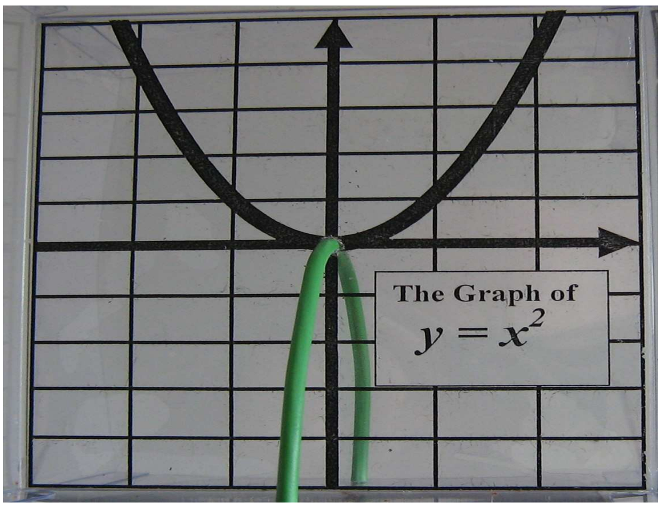
If $y = -9$ then $x^2 = -9$ and $x = \pm 3i$

Thinking very laterally, I thought that instead of just having a ***y axis*** and an ***x AXIS*** (as shown in **Fig 1**) we should have a ***y axis*** but a **complex *x* PLANE!** (as shown on **Fig 2**)

This means that the usual form of the parabola $y = x^2$ exists in the normal ***x, y*** plane **but another part of the parabola exists at right angles to the usual graph.**

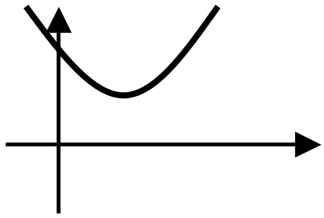
Fig 3 is a Perspex model of $y = x^2$ and its “phantom” hanging at right angles to it.

Fig 3



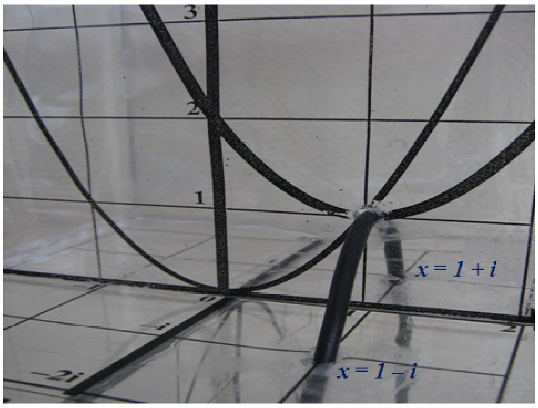
"PHANTOM GRAPHS". Now let us consider the graph $y = (x-1)^2 + 1 = x^2 - 2x + 2$

The minimum real y value is normally thought to be $y = 1$ but now we can have any real y values!



If $y = 0$ then $(x-1)^2 + 1 = 0$
so that $(x-1)^2 = -1$
producing $x - 1 = \pm i$
therefore $x = 1 + i$ *and* $x = 1 - i$

Fig 4

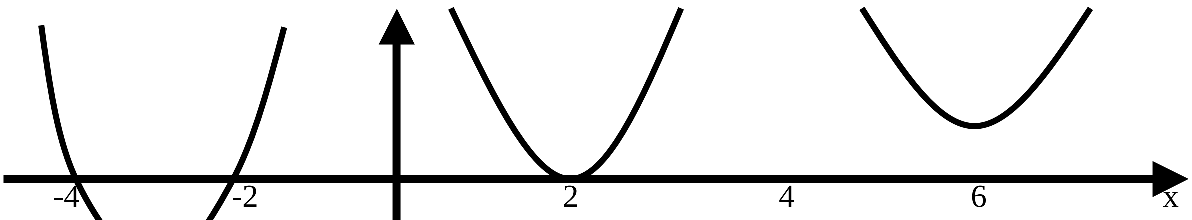


If $y = -3$ then $(x-1)^2 + 1 = -3$
so that $(x-1)^2 = -4$
therefore $x = 1 + 2i$ *and* $x = 1 - 2i$

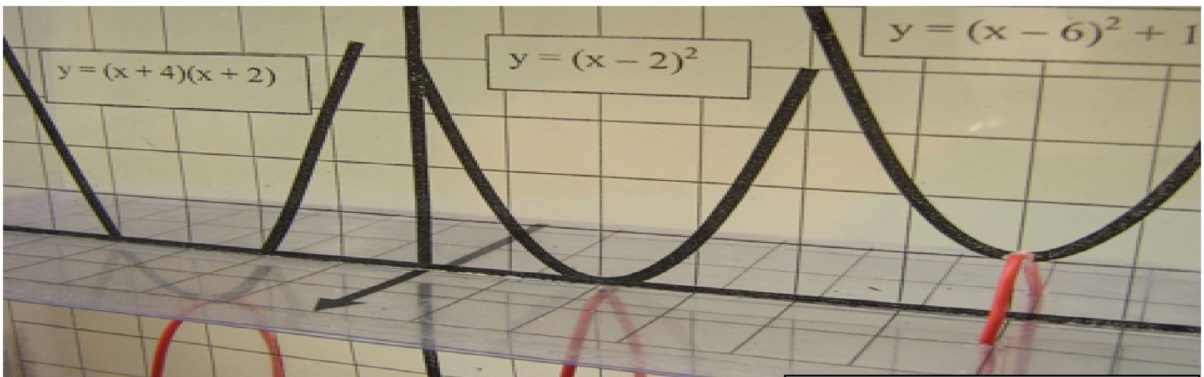
Similarly if $y = -8$ then $(x-1)^2 + 1 = -8$
so that $(x-1)^2 = -9$
therefore $x = 1 + 3i$ *and* $x = 1 - 3i$

The result is another "phantom" parabola which is "hanging" from the normal graph $y = x^2 - 2x + 2$ and the exciting and fascinating part is that the **solutions of** $x^2 - 2x + 2 = 0$ are $1 + i$ and $1 - i$ which are **where the graph crosses the x plane**! See **Fig 4**

In fact ALL parabolas have these "phantom" parts hanging from their lowest points and at right angles to the normal ***x, y*** plane.
It is interesting to consider the 3 types of solutions of quadratics.
Consider these cases: $y = (x+4)(x+2)$; $y = (x-2)^2$; $y = (x-6)^2 + 1$



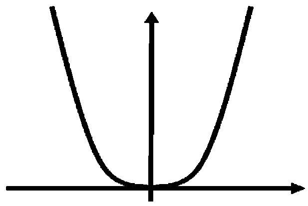
Now if we imagine each graph with its "phantom" hanging underneath we get the full graphs as shown below. **Fig 5**



Here the "phantom" has no effect on the solutions $x = -4$ *and* $x = -2$.	Notice that the curve goes through the point $x = 2$ twice! (a double solution)	The solutions are where the graph crosses the ***x plane*** at $x = 6 \pm i$ (**a conjugate pair**)

Now consider the graph of $y = x^4$

We normally think of this as just a U shaped curve as shown.
This consists of points (0, 0), (±1, 1), (±2, 16), (±3, 81) etc
The fundamental theorem of algebra tells us that equations of the form $x^4 = c$ should have 4 solutions not just 2 solutions.

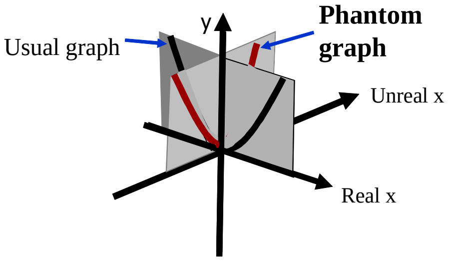


If $y = 1$, $x^4 = 1$ so using De Moivre's Theorem: $r^4 cis\ 4\theta = 1cis\ (360n)$
$r = 1$ and $4\theta = 360n$ therefore $\theta = 0, 90, 180, 270$ producing the 4 solutions :
$x_1 = 1\ cis\ 0 = 1$, $x_2 = 1\ cis\ 90 = i$, $x_3 = 1\ cis\ 180 = -1$ and $x_4 = 1\ cis\ 270 = -i$

If $y = 16$, $x^4 = 16$ so using De Moivre's Theorem: $r^4 cis\ 4\theta = 16cis\ (360n)$
$r = 2$ and $4\theta = 360n$ therefore $\theta = 0, 90, 180, 270$ producing the 4 solutions :
$x_1 = 2\ cis\ 0 = 2$, $x_2 = 2\ cis\ 90 = 2i$, $x_3 = 2\ cis\ 180 = -2$, $x_4 = 2\ cis\ 270 = -2i$

Fig 6

This means $y = x^4$ has another **phantom** part at right angles to the usual graph.



The points (1, 1), (–1, 1), (2, 16), (–2, 16) will produce the ordinary graph but the points (i, 1), (–i, 1), (2i, 16), (–2i, 16) will produce a similar curve at right angles to the ordinary graph. **Fig 6**

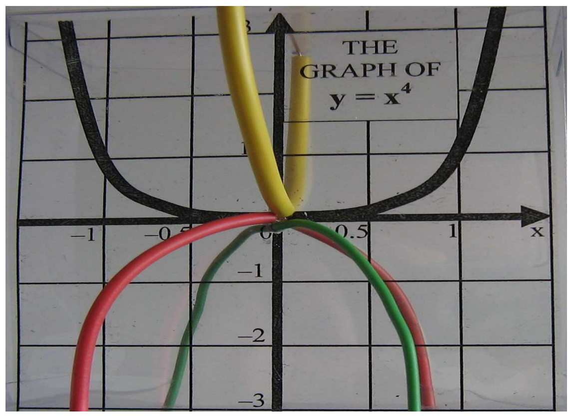
But this is not all!

We now consider **negative** real y values!

Consider $y = -1$ so $x^4 = -1$
Using De Moivre's Theorem:
$r^4 cis\ 4\theta = 1cis\ (180 + 360n)$
$r = 1$ and $4\theta = 180 + 360n$ so $\theta = 45 + 90n$
$x_1 = 1\ cis\ 45$, $x_2 = 1\ cis\ 135$,
$x_3 = 1\ cis\ 225$, $x_4 = 1\ cis\ 315$

Similarly, if $y = -16$, $x^4 = -16$
Using De Moivre's Theorem:
$r^4 cis\ 4\theta = 16cis\ (180 + 360n)$
$r = 2$ and $4\theta = 180 + 360n$ so $\theta = 45 + 90n$
$x_1 = 2\ cis\ 45$, $x_2 = 2\ cis\ 135$,
$x_3 = 2\ cis\ 225$, $x_4 = 2\ cis\ 315$

Fig 7 (photo of Perspex model)

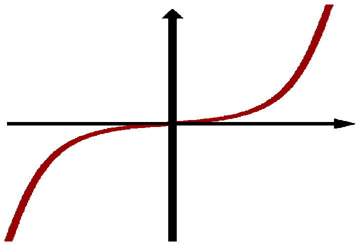


The points corresponding to negative y values produce two curves identical in shape to the two curves for positive y values but they are rotated 45 degrees as shown on **Fig 7.**

NOTE: Any horizontal plane crosses the curve in 4 places because all equations of the form $x^4 = \pm c$ have **4** solutions and it is clear from the photo that the solutions are conjugate pairs!

Consider the basic cubic curve $y = x^3$.

Equations with x^3 have 3 solutions.
If $y = 1$ then $x^3 = 1$
so $r^3 cis\ 3\theta = 1 cis\ (360n)$
$r = 1$ and $\theta = 120n = 0, 120, 240$
$x_1 = 1\ cis\ 0,\ x_2 = 1\ cis\ 120,\ x_3 = 1\ cis\ 240$

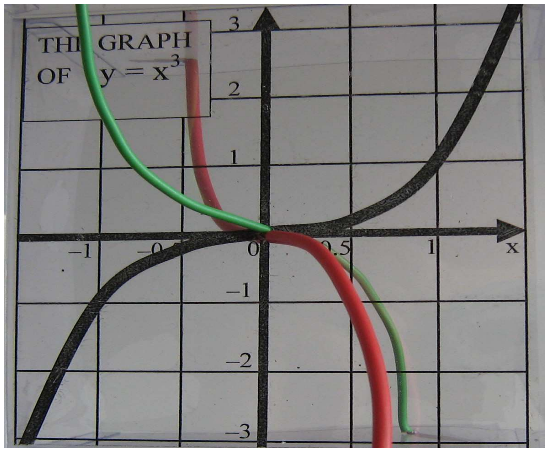


Similarly if $y = 8$ then $x^3 = 8$
so $r^3 cis\ 3\theta = 8 cis\ (360n)$
$r = 2$ and $\theta = 120n = 0, 120, 240$
$x_1 = 2\ cis\ 0,\ x_2 = 2\ cis\ 120,\ x_3 = 2\ cis\ 240$

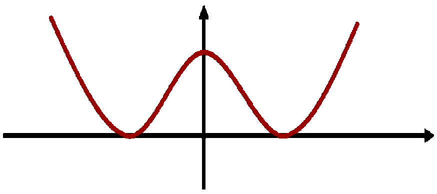
Also y can be negative. If $y = -1$, $x^3 = -1$
so $r^3 cis\ 3\theta = 1 cis\ (180 + 360n)$
$r = 1$ and $3\theta = 180 + 360n$ so $\theta = 60 + 120n$
$x_1 = 1\ cis\ 60,\ x_2 = 1\ cis\ 180,\ x_3 = 1\ cis\ 300$

The result is THREE identical curves situated at 120 degrees to each other! (See Fig 8)

Fig 8 (photo of Perspex model)



Now consider the graph $y = (x + 1)^2(x - 1)^2 = (x^2 - 1)(x^2 - 1) = x^4 - 2x^2 + 1$



Any horizontal line (or plane) should cross this graph at 4 places because any equation of the form $x^4 - 2x^2 + 1 = C$ (where C is a constant) has 4 solutions.

If $x = \pm 2$ then $y = 9$
so solving $x^4 - 2x^2 + 1 = 9$
we get : $x^4 - 2x^2 - 8 = 0$
so $(x + 2)(x - 2)(x^2 + 2) = 0$
giving **$x = \pm 2$ and $\pm\sqrt{2}\, i$**

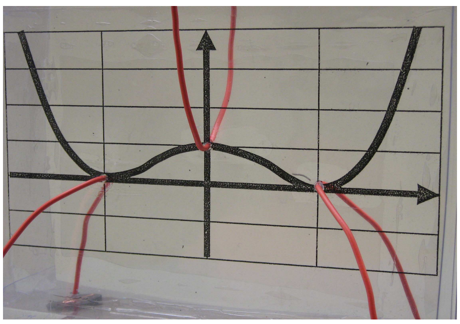
Similarly if $x = \pm 3$ then $y = 64$
so solving $x^4 - 2x^2 + 1 = 64$
we get $x^4 - 2x^2 - 63 = 0$
so $(x + 3)(x - 3)(x^2 + 7) = 0$
giving **$x = \pm 3$ and $\pm\sqrt{7}\, i$**

The complex solutions are all of the form **0** ± n***i***. This means that a **phantom curve**, **at right angles** to the basic curve, stretches upwards from the maximum point.

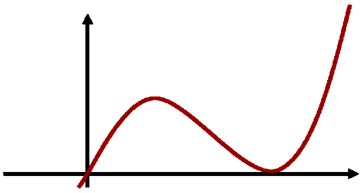
If $y = -1$, $x = -1.1 \pm 0.46i$, $1.1 \pm 0.46i$
If $y = -2$, $x = -1.2 \pm 0.6i$, $1.2 \pm 0.6i$
If $y = -4$, $x = -1.3 \pm 0.78i$, $1.3 \pm 0.78i$

Notice that the real parts of the x values vary. This means that the **phantom** curves hanging off from the two minimum points are not in a vertical plane as they were for the parabola. See **Fig 9. Clearly all complex solutions to $x^4 - 2x^2 + 1 = C$ are conjugate pairs.**

Fig 9 (photo of Perspex model)

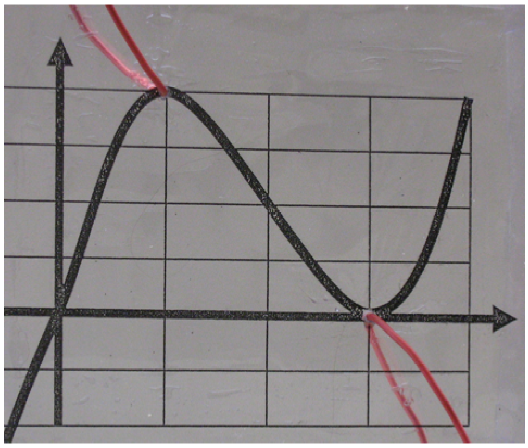


Consider the cubic curve $y = x(x-3)^2$



As before, any horizontal line (or plane) should cross this graph at 3 places because any equation of the form: $x^3 - 6x^2 + 9x =$ ***"a constant"***, has 3 solutions.

Fig 10 (photo of Perspex model)

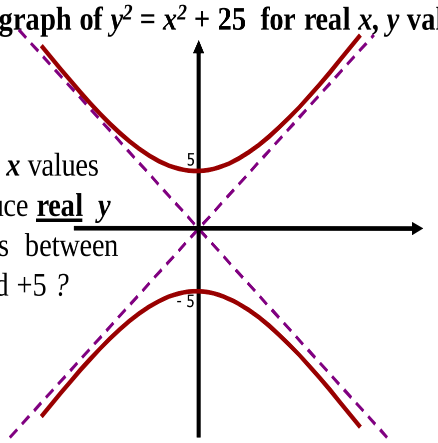


If $x^3 - 6x^2 + 9x = 5$ ***then*** $x = 4.1$ and $\underline{\mathbf{0.95} \pm 0.6\boldsymbol{i}}$
If $x^3 - 6x^2 + 9x = 6$ ***then*** $x = 4.2$ and $\underline{\mathbf{0.90} \pm 0.8\boldsymbol{i}}$
If $x^3 - 6x^2 + 9x = 7$ ***then*** $x = 4.3$ and $\underline{\mathbf{0.86} \pm 0.9\boldsymbol{i}}$
So the left hand phantom is leaning to the left from the maximum point (1, 4).

If $x^3 - 6x^2 + 9x = -1$ ***then*** $x = -0.1$ and $\underline{\mathbf{3.05} \pm 0.6\boldsymbol{i}}$
If $x^3 - 6x^2 + 9x = -2$ ***then*** $x = -0.2$ and $\underline{\mathbf{3.1} \pm 0.8\boldsymbol{i}}$
If $x^3 - 6x^2 + 9x = -3$ ***then*** $x = -0.3$ and $\underline{\mathbf{3.14} \pm 0.9\boldsymbol{i}}$
So the right hand phantom is leaning to the right from the minimum point (3, 0). See Fig 10

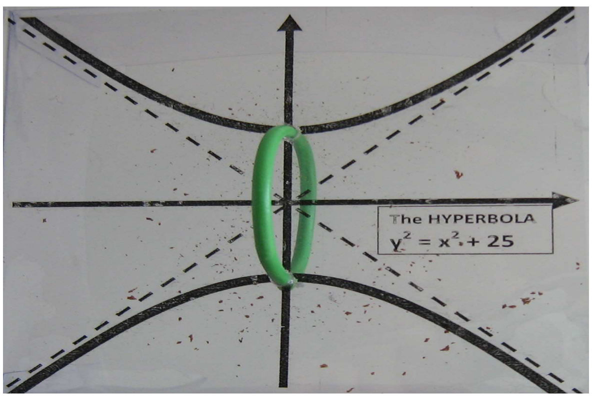
The HYPERBOLA $\underline{y^2 = x^2 + 25.}$ This was the most surprising and absolutely delightful Phantom Graph that I found whilst researching this concept.

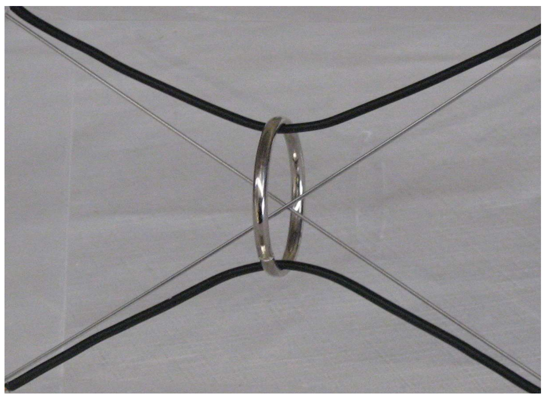
The graph of $y^2 = x^2 + 25$ for real x, y values.



What x values produce **real** y values between -5 and +5 ?

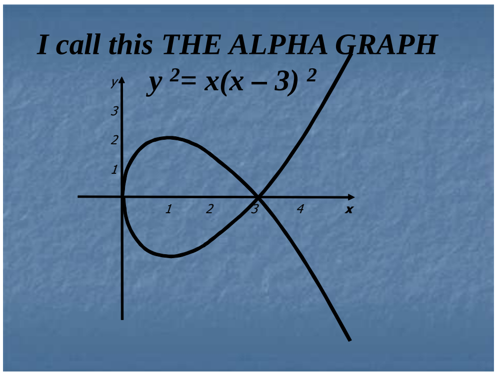
If $y = 4$ then $16 = x^2 + 25$
and $-9 = x^2$
so $x = \pm 3i$
Similarly if $y = 3$ then $9 = x^2 + 25$
so $x = \pm 4i$
And if $y = 0$ then $0 = x^2 + 25$
so $x = \pm 5i$
These are points on a circle of radius 5 units.
(0, 5) ($\pm 3\boldsymbol{i}$, 4) ($\pm 4\boldsymbol{i}$, 3) ($\pm 5\boldsymbol{i}$, 0)
The circle has complex x values but real y values.
This circle is in the plane at right angles to the hyperbola and joining its two halves!
See photos below of the Perspex models.

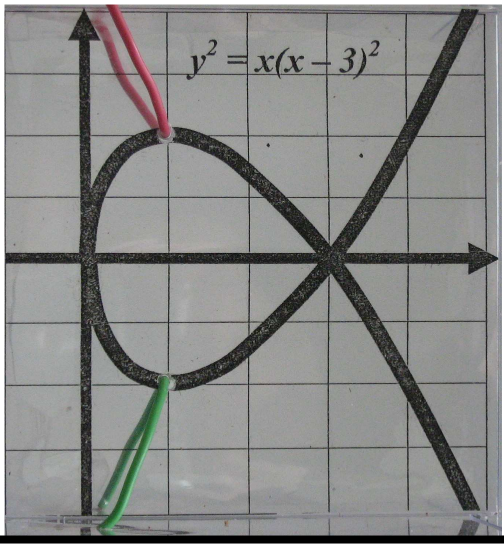




AFTERMATH!!!

I recently started to think about other curves and thought it worthwhile to include them.





Using a technique from previous graphs:
I choose an x value such as $x = 5$,
calculate the y^2 value, ie $y^2 = 20$ and $y \approx 4.5$
then solve the equation $x(x-3)^2 = 20$
already knowing one factor is $(x-5)$

$$\text{ie} \quad x(x-3)^2 = 20$$
$$x^3 - 6x^2 + 9x - 20 = 0$$
$$(x-5)(x^2 - x + 4) = 0$$
$$x = 5 \text{ or } \tfrac{1}{2} \pm 1.9i$$

This means (5, 4.5) is an "ordinary" point on the graph but two "phantom" points are ($\tfrac{1}{2} \pm 1.9i$, 4.5)

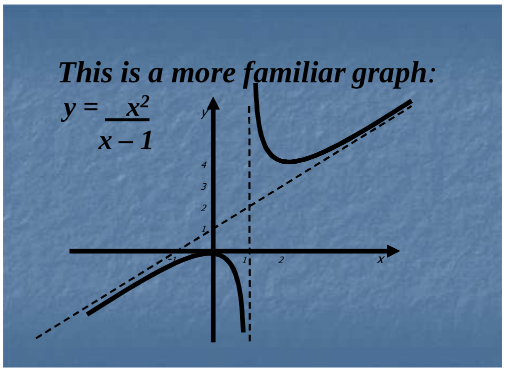
Similarly:
If $x = 6$, $y^2 = 54$ and $y = \pm 7.3$ so $x(x-3)^2 = 54$

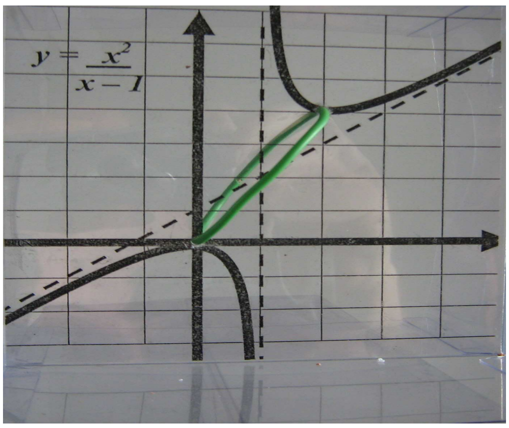
$$x^3 - 6x^2 + 9x - 54 = 0$$
$$(x-6)(x^2 + 9) = 0$$
$$x = 6 \text{ or } \pm 3i$$

And
If $x = 7$, $y^2 = 112$ and $y = \pm 10.6$ so $x(x-3)^2 = 112$

$$x^3 - 6x^2 + 9x - 112 = 0$$
$$(x-7)(x^2 + x + 16) = 0$$
$$x = 7 \text{ or } -\tfrac{1}{2} \pm 4i$$

Hence we get the two phantom graphs as shown.





$$y = \frac{x^2}{x-1}$$

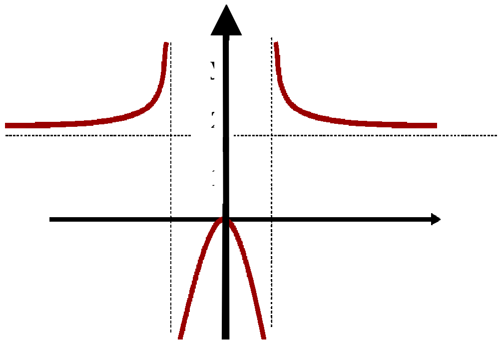
Here we need to find complex x values which produce real y values from 0 to 4.

$$\left.\begin{array}{ll} \text{If } y = 0 & x = 0 \\[2ex] \text{If } y = 1 & x = \frac{1}{2} \pm \frac{\sqrt{3}i}{2} \\[2ex] \text{If } y = 2 & x = 1 \pm i \\[2ex] \text{If } y = 3 & x = \frac{3}{2} \pm \frac{\sqrt{3}i}{2} \\[2ex] \text{If } y = 4 & x = 2 \end{array}\right\}$$

These points produce the phantom "oval" shape as shown in the picture on the left.

Consider the graph $y = \dfrac{2x^2}{x^2 - 1} = 2 + \dfrac{2}{x^2 - 1}$

This has a horizontal asymptote $y = 2$
and two vertical asymptotes $x = \pm 1$



If $y = 1$ *then* $\dfrac{2x^2}{x^2 - 1} = 1$

so $2x^2 = x^2 - 1$

and $x^2 = -1$

producing $x = \pm i$

This implies there is a "phantom graph" which approaches the horizontal asymptotic plane y = 2 and is at right angles to the x, y plane, resembling an upside down normal distribution curve.

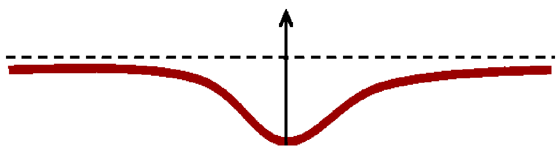
If $y = 1.999$ *then* $\dfrac{2x^2}{x^2 - 1} = 1.999$

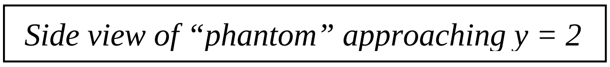
so $2x^2 = 1.999x^2 - 1.999$

and $0.001x^2 = -1.999$

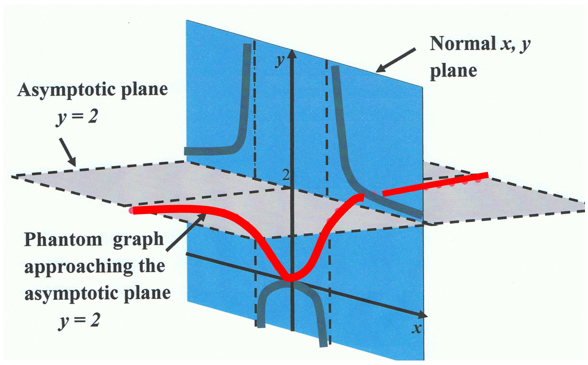
Producing $x^2 = -1999$

$x \approx \pm 45\, i$



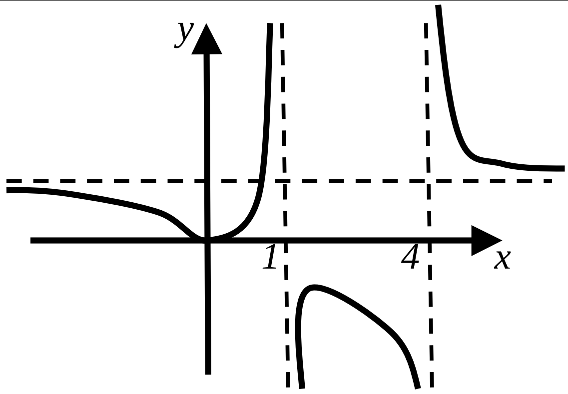


Side view of "phantom" approaching y = 2



Consider an apparently "similar" equation but with a completely different "Phantom".

$y = \dfrac{x^2}{(x-1)(x-4)} = \dfrac{x^2}{x^2 - 5x + 4}$



The minimum point is (0, 0)
The maximum point is (1.6, –1.8)

If $y = -0.1$, $x = 0.2 \pm 0.56i$

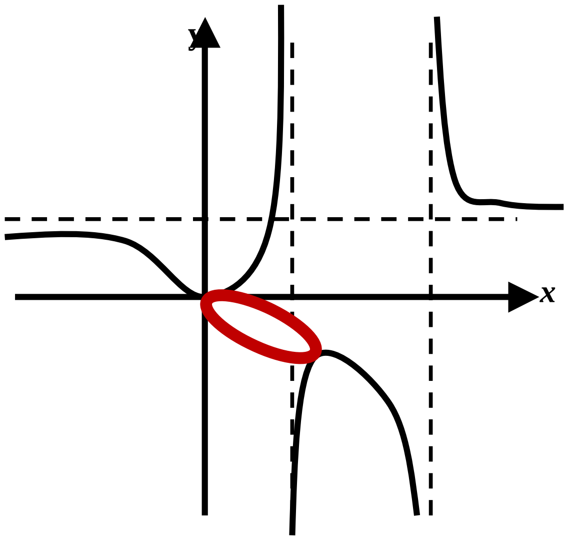
If $y = -0.2$, $x = 0.4 \pm 0.7i$

If $y = -0.5$, $x = 0.8 \pm 0.8i$

If $y = -1$, $x = 1.25 \pm 0.66i$

If $y = -1.5$, $x = 1.5 \pm 0.4i$

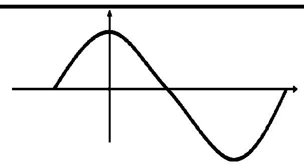
If $y = -1.7$, $x = 1.6 \pm 0.2i$



These results imply that a "phantom" ***oval*** *shape joins the minimum point (0, 0) to the maximum point (1.6, –1.78).*

The final two graphs I have included in this paper involve some theory too advanced for secondary students but I found them absolutely fascinating!

If $y = \cos(x)$ what about y values > 1 and < - 1 ?



Using $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots\ldots.$

Let's find $\cos(\pm\, i) = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \frac{1}{8!} \ldots\ldots..$

$\approx$ **1.54 (ie > 1)**

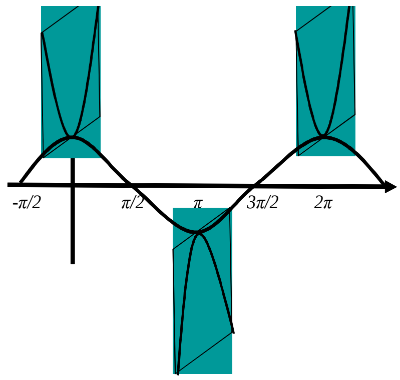
Similarly $\cos(\pm\, 2i) = 1 + \frac{4}{2!} + \frac{16}{4!} + \frac{64}{6!} + \ldots$

$\approx$ **3.8**

Also find $\cos(\pi + i) = \cos(\pi)\cos(i) - \sin(\pi)\sin(i)$

$= -1 \times \cos(i) - 0$

$\approx$ **– 1.54 (ie < –1)**



These results imply that the cosine graph also has its own "**phantoms**" in vertical planes at right angles to the usual ***x, y*** graph, emanating from each max/min point.

Finally consider the exponential function $y = e^x$. How can we find x if $e^x = -1$?

Using the expansion for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\ldots$

We can find $e^{xi} = 1 + xi + \frac{(xi)^2}{2!} + \frac{(xi)^3}{3!} + \frac{(xi)^4}{4!} + \frac{(xi)^5}{5!} + \ldots\ldots$

$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$

$= \quad (\cos x) \quad + \quad i\,(\sin x)$

If we are to get **REAL y values** then using $e^{xi} = \cos x + i \sin x$, we see that **sin *x*** must be zero.

This only occurs when $x = 0, \pi, 2\pi, 3\pi, \ldots$ (or generally $n\pi$)

$e^{\pi i} = \cos\pi + i\sin\pi = -1 + 0i$, $e^{2\pi i} = \cos 2\pi + i\sin 2\pi = +1 + 0i$

$e^{3\pi i} = \cos 3\pi + i\sin 3\pi = -1 + 0i$, $e^{4\pi i} = \cos 4\pi + i\sin 4\pi = +1 + 0i$

Now consider $y = e^X$ where $X = x + 2n\pi i$ (ie even numbers of π)

ie $y = e^{x + 2n\pi i} = e^x \times e^{2n\pi i} = e^x \times 1 = e^x$

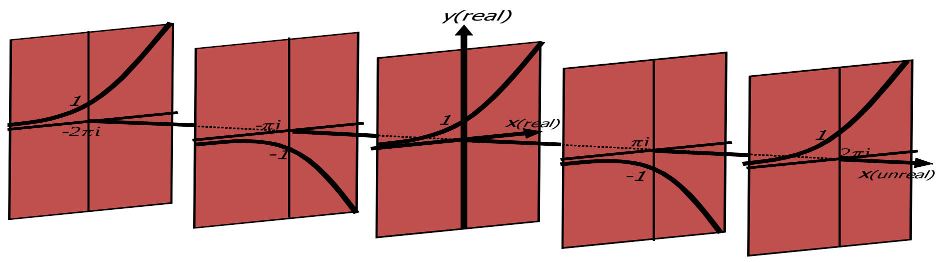
Also consider $y = e^X$ where $X = x + (2n+1)\pi i$ (ie odd numbers of π)

ie $y = e^{x + (2n+1)\pi i} = e^x \times e^{(2n+1)\pi i} = e^x \times -1 = -e^x$

This means that the graph of $y = e^X$ consists of **parallel identical curves** if $X = x + 2n\pi i$

$= x$ + even N[os] of πi

and, **upside down parallel identical curves** occurring at $X = x + (2n+1)\pi i = x$ + odd N[os] of πi



Graph of $y = e^X$ where $X = x + n\pi i$