Exploring differential geometric space using GeoGebra

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ABSTRACT. *In this paper, we explore differential geometric space with curves on 3-dimensional environment constructed in GeoGebra. Firstly, we construct 3-dimensional environment in GeoGebra and Frenet-Serret frame on a curve. Next we explore some curves in space and curves mapped on surface.*

1 Introduction

Park *et al*.(2010) constructed 3-dimensional environment in GeoGebra for 3-dimensional graphs of functions. In this environment, curves and surfaces can be represented and rotated in 3-dimensional space. Although only 2 coordinates can be used in GeoGebra, 3 coordinates can be represented using new basis defined (V_1, V_2, V_3). Moreover, two vectors can be operated by sum of vectors and scalar multiplication in this environment.

In this paper, I will construct environment for exploring the differential geometry of curves and surfaces in 3-dimensional space. Firstly, I will introduce Park *et al*.(2010)'s result. Secondly, Frenet-Serret frame for a curve in GeoGebra will be constructed and the variation of vectors (T, N, B) be observed. Thirdly, curves on a surface will be observed. In the process of construction, the operations of vector, cross product and inner product, should be defined. I will define cross product and inner product using relative cell reference functionality in spreadsheet view.

Now I will start next section with introducing functionality of GeoGebra for representing graphs.

2 Constructing basis in 3-dimensional space

2.1 Functionality of GeoGebra

GeoGebra is an educational software which can manipulate 2-dimensional mathematical objects with algebraic and geometric representation. For example, y = 2x + 1, a linear

function, is represented an equation in algebra view and a line in geometry view. GeoGebra also has command and slider. Slider is the visualization of variable in GeoGebra.

For example, after making slider a, we can type (a, - a + 1) in input field in order to make a point on

$y = -x + 1$ in geometry view of GeoGebra.

Generally, standard basis B in 3-dimensional space is $\{(1,0,0)^t, (0,1,0)^t, (0,0,1)^t\}$. In GeoGebra, each column vectors, E_1, E_2, E_3 can be defined repectively.

```
E_1 = {{1},{0},{0}}
E_2 = {{0},{1},{0}}
E_3 = {{0},{0},{1}}
```

Then three basic rotation matrices($R_x(a)$, $R_y(b)$, $R_z(c)$) around x-axis, y-axis and z-axis can be defined respectively using three sliders, a, b and c.

```
R_x = {{1,0,0},{0,cos(a),-sin(a)},{0,sin(a),cos(a)}}
R_y = {{cos(b),0,-sin(b)},{0,1,0},{sin(b),0,cos(b)}}
R_z = {{cos(c),-sin(c),0},{sin(c),cos(c),0},{0,0,1}}
```

Multiplying three basic matrix, $R_x(a)$, $R_y(b)$ and $R_z(c)$, we can get the rotation matrix $R_{xyz}(a, b, c)$.

R_{xyz} = R_{x}*R_{y}*R_{z}

Now, we multiply $R_{xyz}(a, b, c)$ to each vectors in standard basis of 3-dimensional space. Then we can get e_1, e_2, e_3 which are vectors rotated by angle a, b and c.

```
e1 = R_{xyz}*E_1
e2 = R_{xyz}*E_2
e3 = R_{xyz}*E_3
```
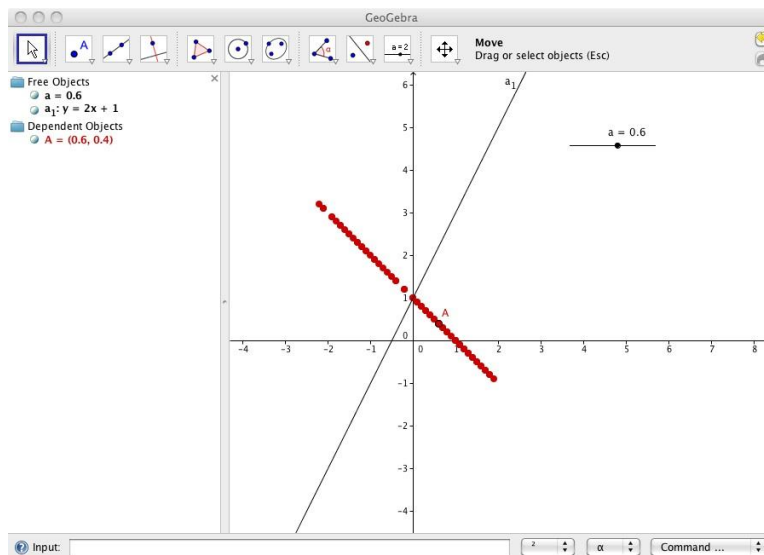

Figure 1: GeoGebra's algebra view and geometry view

Then we can get basis B_{ROT} = {e_1, e_2, e_3} rotated by *a*, *b* and *c*. However, the vectors, e_1, e_2, e_3, has 3 coordinates, we should project each vectors to *yz*-plane for representing in geometry view of GeoGebra. Finally, we can get vectors, e_1, e_2, e_3, which represent 3-dimenstional basis projected into 2-dimensional space (e_1, e_2 and e_3 are vectors and V_1, V_2 and V_3 are points).

```
e_1=(Element[Element[e1,2],1], Element[Element[e1,3],1])
e_2=(Element[Element[e2,2],1], Element[Element[e2,3],1])
e_3=(Element[Element[e3,2],1], Element[Element[e3,3],1])
V_1 = e_1
V_2 = e_2
V_3 = e_3
```

2.3   Applications

Before starting this section, we added some decorations in geometry view, *x*-axis, *y*-axis, *z*-axis and each plane, which can help the figures recognized well in 3-dimensional space.

2.3.1   Tetrahedron

The four vertices of tetrahedron are P_1 = (0,0,0), P_2=(2,0,0), P_3=(1,$\sqrt{3}$,0), P_4 = (1, $\sqrt{3}/3$, $2\sqrt{6}/3$) and we can type these commands in input field of GeoGebra (Figure 3).

```
P_1 = 0 V_1 + 0 V_2 + 0 V_3
P_2 = 2 V_1 + 0 V_2 + 0 V_3
P_3 = 1 V_1 + sqrt(3) V_2 + 0 V_3
P_4 = 1 V_1 + sqrt(3)/3 V_2 + 2*sqrt(6)/3 V_3
```
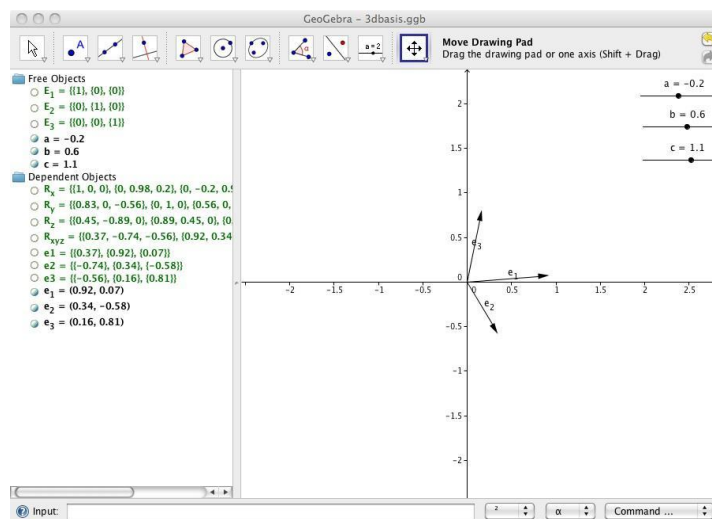

Figure 2: 3-dimensional basis constructed in GeoGebra

2.3.2　Helix

Next we will draw helix in 3-dimensional space. We can parameterize a point on a helix as $(\cos(t), \sin(t), t)$.
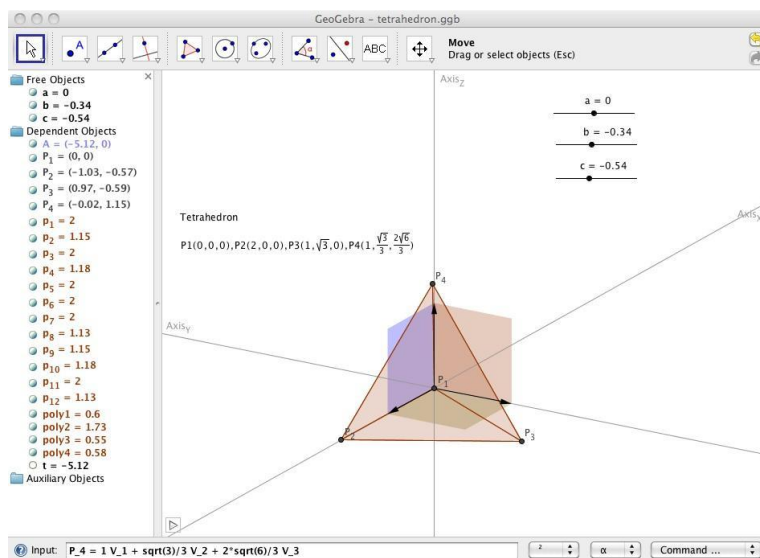
```
E = cos(t) V_1 + sin(t) V_2 + t V_3
```


Figure 3: Tetrahedron in 3-dimensional space

In this time, t is already defined as *x*-coordinate value of point A on *x*-axis (2-dimensional environment of GeoGebra). We can draw *locus* of the point using locus command/tool, if we type locus[E, A] in input field of GeoGebra.

2.3.3　Surfaces

We can't draw surfaces directly in geometry view of GeoGebra; it is impossible to fill colors in arbitrary Jordan curves (simple closed curves). I found some alternative solutions, drawing contours and mapping some lines from *xy*-plane (domain) on the surface.

Firstly, we choose five points of the same z-coordinate value. Then, we define a quadratic curve(conic section) with five points, as we can make a curve(contour) with five points chosen using Conic through Five Points tool of GeoGebra.

```
P_1 = sqrt(1 + 1/u) V_1 + u V_3
P_2 = sqrt(1 + 1/u) V_2 + u V_3
P_3 = -sqrt(1 + 1/u) V_2 + u V_3
P_4 = -sqrt(1 + 1/u) V_1 + u V_3
P_5 = (sqrt(1 + 1/u) V_1 + sqrt(1 + 1/u) V_2)/sqrt(2) + u
V_3
```
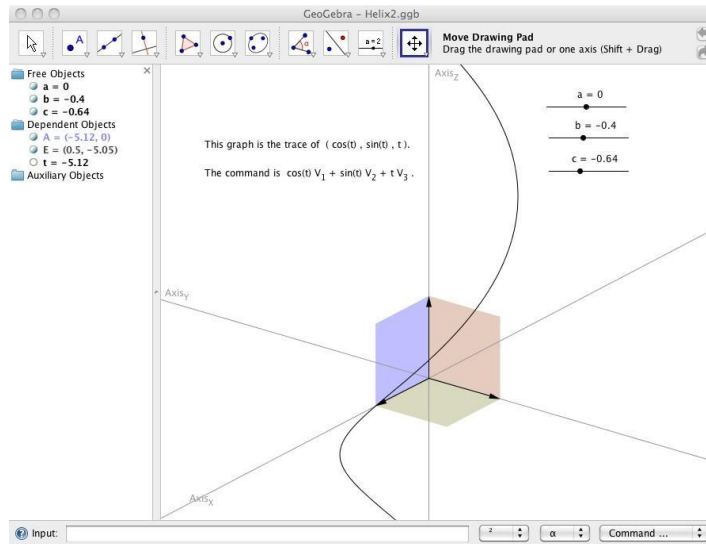
Figure 4: Helix in 3-dimensional space

Secondly, in *xy*-plane (domain), we make points go through various paths, especially line paths. We can map these lines through a function into 3-dimensional space using locus command/tool and spreadsheet view in GeoGebra. In spreadsheet view, we can create the rigid on *xy*-plane and map them into 3-dimensional space (Figure 6, Figure 7).
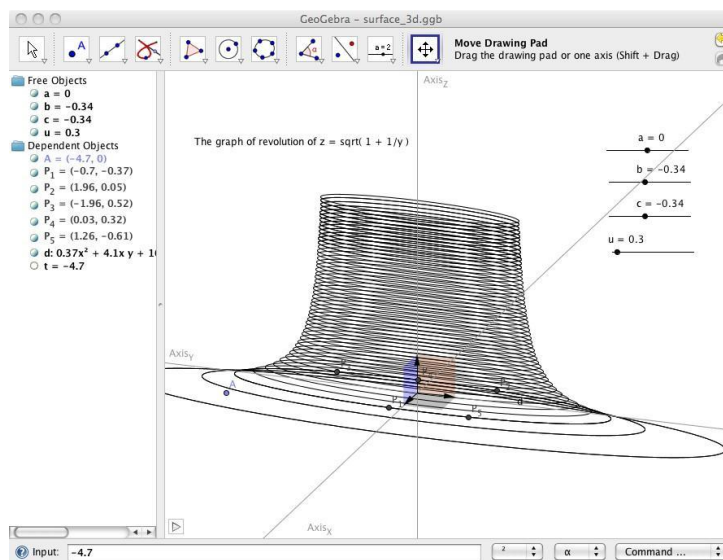

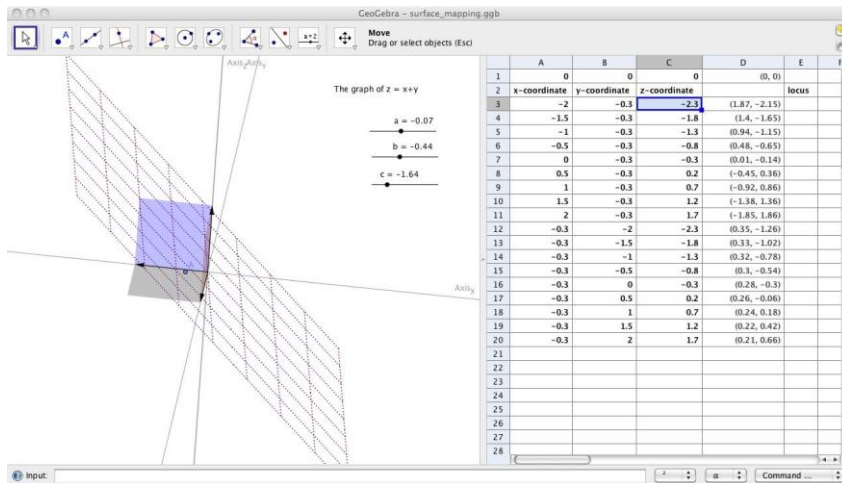**Figure 5**: Surface of revolution ($z = \sqrt{1 + 1/y}$)

The graph of z = x+y

a = -0.07

b = -0.44

c = 1.64

	A	B	C	D	E	F
1	0	0	0	(0, 0)		
2	x-coordinate	y-coordinate	z-coordinate	locus		
3	-2	-0.3	-2.3	(1.87, -2.15)		
4	-1.5	-0.3	-1.8	(1.4, -1.65)		
5	-1	-0.3	-1.3	(0.94, -1.15)		
6	-0.5	-0.3	-0.8	(0.48, -0.65)		
7	0	-0.3	-0.3	(0.01, -0.14)		
8	0.5	-0.3	0.2	(-0.45, 0.36)		
9	1	-0.3	0.7	(-0.92, 0.86)		
10	1.5	-0.3	1.2	(-1.38, 1.36)		
11	2	-0.3	1.7	(-1.85, 1.86)		
12	-0.3	-2	-2.3	(0.35, -1.26)		
13	-0.3	-1.5	-1.8	(0.33, -1.02)		
14	-0.3	-1	-1.3	(0.32, -0.78)		
15	-0.3	-0.5	-0.8	(0.3, -0.54)		
16	-0.3	0	-0.3	(0.28, -0.3)		
17	-0.3	0.5	0.2	(0.26, -0.06)		
18	-0.3	1	0.7	(0.24, 0.18)		
19	-0.3	1.5	1.2	(0.22, 0.42)		
20	-0.3	2	1.7	(0.21, 0.66)		

Figure 6: Graph of $z = x + y$

The graph of z = x^2 + y^2

a = -0.07

b = -0.44

c = -0.44

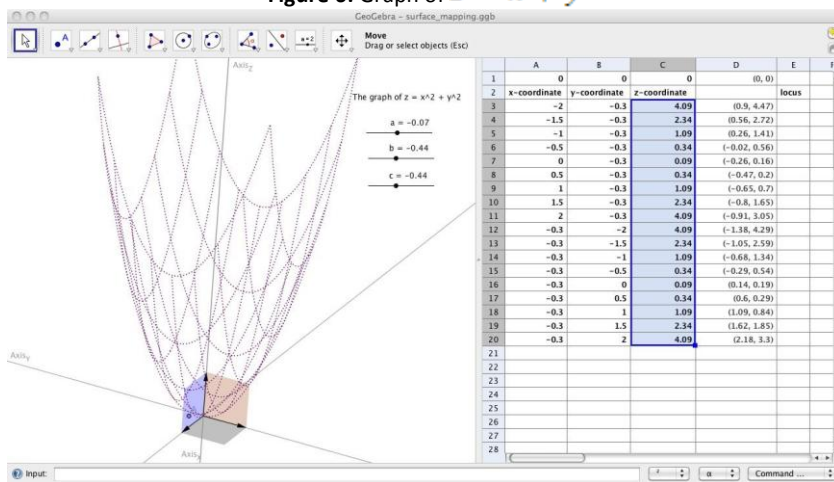|	A	B	C	D	E	F
1	0	0	0	(0, 0)		
2	x-coordinate	y-coordinate	z-coordinate	locus		
3	-2	-0.3	4.09	(0.9, 4.47)		
4	-1.5	-0.3	2.34	(0.56, 2.72)		
5	-1	-0.3	1.09	(0.26, 1.41)		
6	-0.5	-0.3	0.34	(-0.02, 0.56)		
7	0	-0.3	0.09	(-0.26, 0.16)		
8	0.5	-0.3	0.34	(-0.47, 0.2)		
9	1	-0.3	1.09	(-0.65, 0.7)		
10	1.5	-0.3	2.34	(-0.8, 1.65)		
11	2	-0.3	4.09	(-0.91, 3.05)		
12	-0.3	-2	4.09	(-1.38, 4.29)		
13	-0.3	-1.5	2.34	(-1.05, 2.59)		
14	-0.3	-1	1.09	(-0.68, 1.34)		
15	-0.3	-0.5	0.34	(-0.29, 0.54)		
16	-0.3	0	0.09	(0.14, 0.19)		
17	-0.3	0.5	0.34	(0.6, 0.29)		
18	-0.3	1	1.09	(1.09, 0.84)		
19	-0.3	1.5	2.34	(1.62, 1.85)		
20	-0.3	2	4.09	(2.18, 3.3)		

Figure 7: Graph of $z = x^2 + y^2$

3. Constructing environment for differential geometry

In this section, we will review some formulas of differential geometry. Then I will construct environment for exploring differential geometry and observe movement of points in space and on a surface.

3.1 Frenet-Serret frame

Let $r(t)$ be a curve in 3-dimensional Euclidean space and $s(t)$ represent the arc length of a point which has moved along $r(t)$ ($r'(t) \neq 0$). In detail, $s(t)$ is given by $s(t) = \int_0^t |r'(\tau)| d\tau$.

With the curve parametrized by its arc length, $r(s) = r(t(s))$, it is possible to define the Frenet-Serret frame (or TNB frame):

· The tangent unit vector T is defined as
$$T = \frac{dr}{ds}$$
· The normal unit vector N is defined as
$$N = \frac{\frac{dT}{ds}}{\left|\frac{dT}{ds}\right|}$$

· The binormal unit vector B is defined as the cross product of T and N:
$$B = T \times N$$

If the curve is not parametrized by arc length, we could use other equivalent expression of Frenet-Serret frame. Suppose that the curve is given by $r(t)$, then the tangent unit vector, T, can be written as

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

The normal unit vector N can be represented as

$$N(t) = \frac{T(t)}{|T'(t)|} = \frac{r'(t) \times (r''(t) \times r'(t))}{|r'(t)| \cdot |r''(t) \times r'(t)|}$$

The binormal unit vector B is then

$$B(t) = T(t) \times N(t) = \frac{r'(t) \times r''(t)}{|r'(t) \times r''(t)|}$$

The curvature κ is

$$\kappa = \frac{r'(t) \times r''(t)}{|r'(t)|^3}$$

And the torsion τ is

$$\tau = \frac{(r'(t) \times r''(t)) \cdot r'''(t)}{|r'(t) \times r''(t)|^2}$$

3.2. Constructing Frenet-Serret frame on a curve

In 3-dimensional environment constructed before, I will define a curve $x(t)$ parametrized by variable t. For defining a curve $x(t) = (x_1(t), x_2(t), x_3(t))$ in 3-dimensional space, we need to define each coordinate functions, x_{1}(x), x_{2}(x), x_{3}(x) and their derivatives using the GeoGebra command, derivative[ ] in advance. For example, I define coordinate functions of helix and its derivatives on GeoGebra as follows.

```
x_1(x) = sin(x)
x_2(x) = cos(x)
x_3(x) = x

dx_1(x) = derivative[x_1(x)]
dx_2(x) = derivative[x_2(x)]
dx_3(x) = derivative[x_3(x)]

ddx_1(x) = derivative[dx_1(x)]
ddx_2(x) = derivative[dx_2(x)]
ddx_3(x) = derivative[dx_3(x)]

dddx_1(x) = derivative[ddx_1(x)]
dddx_2(x) = derivative[ddx_2(x)]
dddx_3(x) = derivative[ddx_3(x)]
```

And a point on helix can be represented as follows (t is already defined).

A = x_1(t) V_1 + x_2(t) V_2 + x_3(t) V_3

For constructing Frenet-Serret frame on a curve, we should define the operations of vector, cross product and inner product in 3-dimensional vector space. Using spreadsheet view in GeoGebra, we can define these operations easily.

In Figure 8, I entered each entries of vector (1, 2, 3) into cell A2, B2, C2 and vector (4, 5, 6) into cell A3, B3, C3 respectively. Then I define A4, B4, C4 and D4 for using relative reference functionality as follows.

```
A4 : B2*C3 - B3*C2
B4 : A3*C2 - A2*C3
C4 : A2*B3 - A3*B2
D4 : A4 V_1 + B4 V_2 + C4 V_3
```

Inner product can be defined similarly in spreadsheet view of GeoGebra (Figure 9).

	A	B	C	D
1	Cross	Product		
2	1	2	3	
3	4	5	6	
4	B2 C3 – B3 C2	A3 C2 – A2 C3	A2 B3 – A3 B2	A4 V_1 + B4 V_2 + C4 V_3

Figure 8: Defining the operation of cross product

	E	F	G	H
	Inner Product			
	1	2	3	
	4	5	6	
	E2 E3	F2 F3	G2 G3	E4 + F4 + G4

Figure 9: Defining the operation of inner product

```
E4 : E2*E3
F4 : F2*F3
G4 : G2*G3
H4 : E4 V_1  + F4 V_2 + G4 V_3
```

Now we can construct Frenet-Serret frame in spreadsheet view as follows (Figure 10).

	A	B	C	D	E	F
6	Curve	VAR : d		Derivatives	Absolute value	Stand...
7	$dx_1(t)$	$dx_2(t)$	$dx_3(t)$	A7 V_1 + B7 V_2 + C7 V_3	sqrt(A7² + B7² + C7²)	D7 / E7
8	$ddx_1(t)$	$ddx_2(t)$	$ddx_3(t)$			
9	B7 C8 – B8 C7	A8 C7 – A7 C8	A7 B8 – A8 B7	A9 V_1 + B9 V_2 + C9 V_3	sqrt(A9² + B9² + C9²)	D9 / E9
10						
11	Binormal Vector					
12	A9 / E9	B9 / E9	C9 / E9			
13	A7 / E7	B7 / E7	C7 / E7			
14	B12 C13 – B13 C12	A13 C12 – A12 C13	A12 B13 – A13 B12	A14 V_1 + B14 V_2 + C14 V_3		

Figure 10: Defining Frenet-Serret frame of $x(t)$

If you fill the cells in spreadsheet view as follows, you can get the tangent unit vector of x(t), T(t), in cell F7.

```
A7 : dx_1(t)
B7 : dx_2(t)
C7 : dx_3(t)
D7 : A7 V_1 + B7 V_2 + C7 V_3
E7 : sqrt(A7^2 + B7^2 + C7^2)
F7 : D7/E7
```

In case of the normal unit vector of *x*(*t*), N(*t*), you can get it in cell F9 when copying and pasting cells from A4 to F4 in row 4 into cell A9 (relative reference functionality of spreadsheet view).

```
A8 : ddx_1(t)
B8 : ddx_2(t)
C8 : ddx_3(t)
Copy and paste A4, B4, C4, D4, E4, F4 into A9
```

We can get the binormal unit vector of x(t), B(t) in the same way (Figure 10).

```
A12 : A9 / E9
B12 : B9 / E9
C12 : C9 / E9

A13 : A7 / E7
B13 : B7 / E7
C13 : C7 / E7
Copy and paste A4, B4, C4, D4 into A14
```
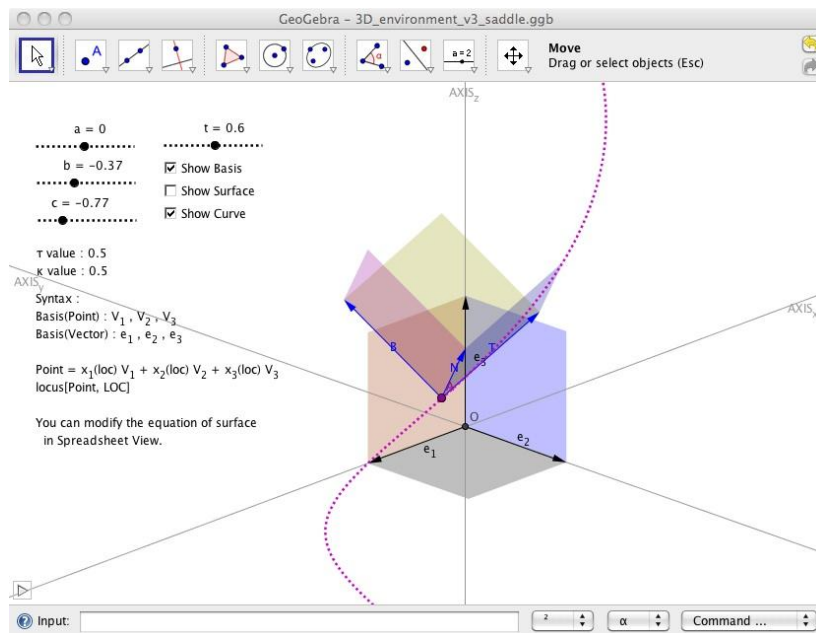

Figure 11: Frenet-Serret frame of curve $x(t)$

3.3. Examples of some curves

In this environment, we can observe many curves and movement of the point on the curves which we haven't seen yet. For example, if we change `x_1(x) = sin(x)` to `1/x^2`, we can get the graph of Figure 12.
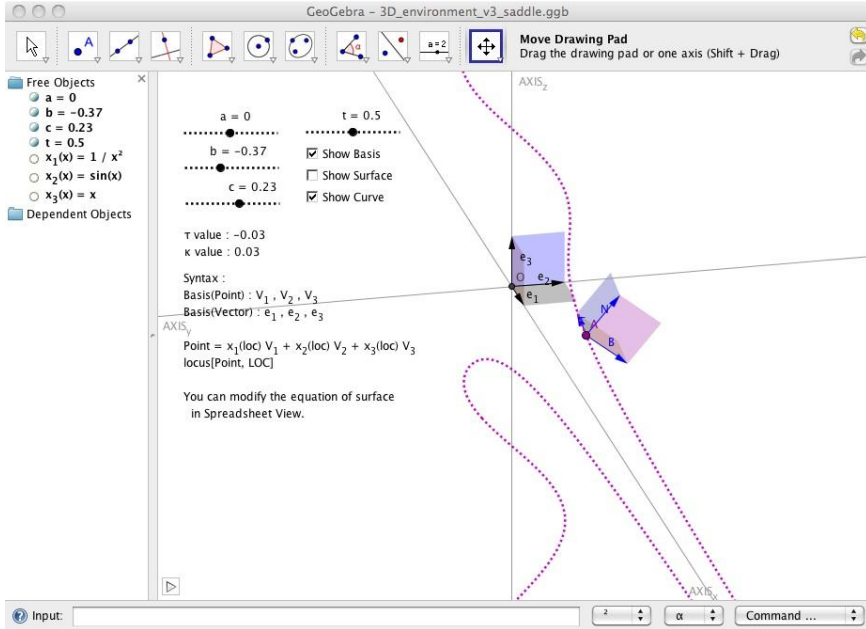If we change `x_1(x)` to `cos(2x)` and `x_2(x)` to `cosh(x)`, we can get the graph of Figure 13.

Figure 12: Graph of $\left(\frac{1}{x^2}, \sin(x), x\right)$
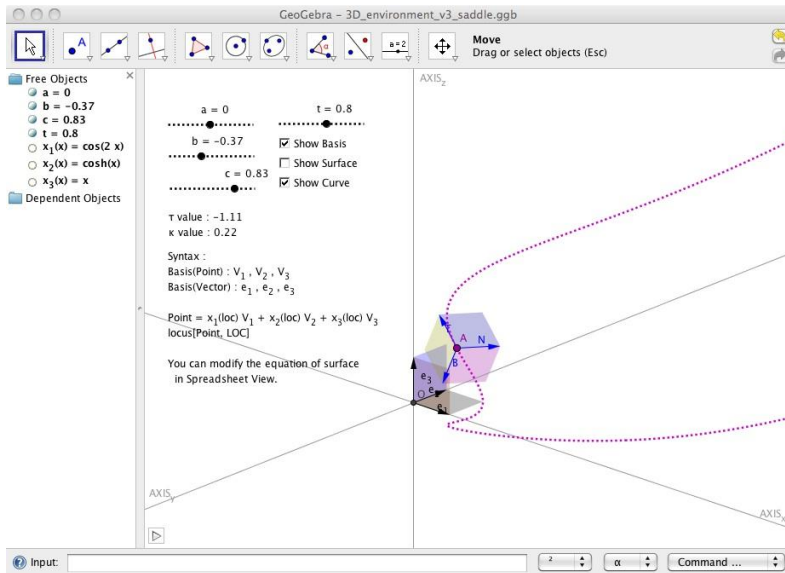

Figure 13: Graph of $(\cos(2x), \cosh(x), x)$

4. Exploring surfaces with curves

In this section, we will explore surfaces with some tools, curves on the surface defined on xy-plane (domain). The movement of the variation vectors, T, N, B and the values of κ and τ will help you observe movement of the curve.

4.1. Mapping curves into the surface

4.1.1. Review of defining surface

In Figure 14, column A and B are used for defining the grid on *xy*-plane and column C for defining the function, $z = -x^2 + y^2$. Using auto completion functionality in spreadsheet view, we can fill the rest of cells in column C easily.

loc_{s} in column A and B is the value of *x*-coordinate of point (LOC_{s}) on *x*-axis in 2-dimensional environment of GeoGebra for *locus* command/tool.

	A	B	C	D	E
21	Grid			POINT	
22	-3	loc_s	$-A22^2 + B22^2$	$A22\ V_1 + B22\ V_2 + C22\ V_3$	Locus[D22, LOC_s]
23	-2.5	loc_s	$-A23^2 + B23^2$	$A23\ V_1 + B23\ V_2 + C23\ V_3$	Locus[D23, LOC_s]
24	-2	loc_s	$-A24^2 + B24^2$	$A24\ V_1 + B24\ V_2 + C24\ V_3$	Locus[D24, LOC_s]
25	-1.5	loc_s	$-A25^2 + B25^2$	$A25\ V_1 + B25\ V_2 + C25\ V_3$	Locus[D25, LOC_s]
26	-1	loc_s	$-A26^2 + B26^2$	$A26\ V_1 + B26\ V_2 + C26\ V_3$	Locus[D26, LOC_s]
27	-0.5	loc_s	$-A27^2 + B27^2$	$A27\ V_1 + B27\ V_2 + C27\ V_3$	Locus[D27, LOC_s]
28	0	loc_s	$-A28^2 + B28^2$	$A28\ V_1 + B28\ V_2 + C28\ V_3$	Locus[D28, LOC_s]
29	0.5	loc_s	$-A29^2 + B29^2$	$A29\ V_1 + B29\ V_2 + C29\ V_3$	Locus[D29, LOC_s]
30	1	loc_s	$-A30^2 + B30^2$	$A30\ V_1 + B30\ V_2 + C30\ V_3$	Locus[D30, LOC_s]
31	1.5	loc_s	$-A31^2 + B31^2$	$A31\ V_1 + B31\ V_2 + C31\ V_3$	Locus[D31, LOC_s]
32	2	loc_s	$-A32^2 + B32^2$	$A32\ V_1 + B32\ V_2 + C32\ V_3$	Locus[D32, LOC_s]
33	2.5	loc_s	$-A33^2 + B33^2$	$A33\ V_1 + B33\ V_2 + C33\ V_3$	Locus[D33, LOC_s]
34	3	loc_s	$-A34^2 + B34^2$	$A34\ V_1 + B34\ V_2 + C34\ V_3$	Locus[D34, LOC_s]
35	loc_s	-3	$-A35^2 + B35^2$	$A35\ V_1 + B35\ V_2 + C35\ V_3$	Locus[D35, LOC_s]
36	loc_s	-2.5	$-A36^2 + B36^2$	$A36\ V_1 + B36\ V_2 + C36\ V_3$	Locus[D36, LOC_s]
37	loc_s	-2	$-A37^2 + B37^2$	$A37\ V_1 + B37\ V_2 + C37\ V_3$	Locus[D37, LOC_s]
38	loc_s	-1.5	$-A38^2 + B38^2$	$A38\ V_1 + B38\ V_2 + C38\ V_3$	Locus[D38, LOC_s]
39	loc_s	-1	$-A39^2 + B39^2$	$A39\ V_1 + B39\ V_2 + C39\ V_3$	Locus[D39, LOC_s]
40	loc_s	-0.5	$-A40^2 + B40^2$	$A40\ V_1 + B40\ V_2 + C40\ V_3$	Locus[D40, LOC_s]
41	loc_s	0	$-A41^2 + B41^2$	$A41\ V_1 + B41\ V_2 + C41\ V_3$	Locus[D41, LOC_s]
42	loc_s	0.5	$-A42^2 + B42^2$	$A42\ V_1 + B42\ V_2 + C42\ V_3$	Locus[D42, LOC_s]
43	loc_s	1	$-A43^2 + B43^2$	$A43\ V_1 + B43\ V_2 + C43\ V_3$	Locus[D43, LOC_s]
44	loc_s	1.5	$-A44^2 + B44^2$	$A44\ V_1 + B44\ V_2 + C44\ V_3$	Locus[D44, LOC_s]
45	loc_s	2	$-A45^2 + B45^2$	$A45\ V_1 + B45\ V_2 + C45\ V_3$	Locus[D45, LOC_s]
46	loc_s	2.5	$-A46^2 + B46^2$	$A46\ V_1 + B46\ V_2 + C46\ V_3$	Locus[D46, LOC_s]
47	loc_s	3	$-A47^2 + B47^2$	$A47\ V_1 + B47\ V_2 + C47\ V_3$	Locus[D47, LOC_s]

Figure 14: locus for representing surface in spreadsheet view

4.1.2. Defining mapping function

We will explore $z = f(x, y)$ type of function (surface) here. The curve constructed before has 3 coordinate functions, `x_1(x)`, `x_2(x)` and `x_3(x)`. If we define `x_3(x)` as the equation of `x_1(x)` and `x_2(x)`, a point on the curve will move on the surface with Frenet-Serret frame. If you want to see the domain curve, define a point of `V_{dom} = x_1(loc) V_1 + x_2(loc) V_2` and type `locus[ V_{dom}, LOC ]` in input

field of GeoGebra. loc is *x*-coordinate value of point LOC defined on *x*-axis in 2-dimensional environment of GeoGebra (Figure 15).
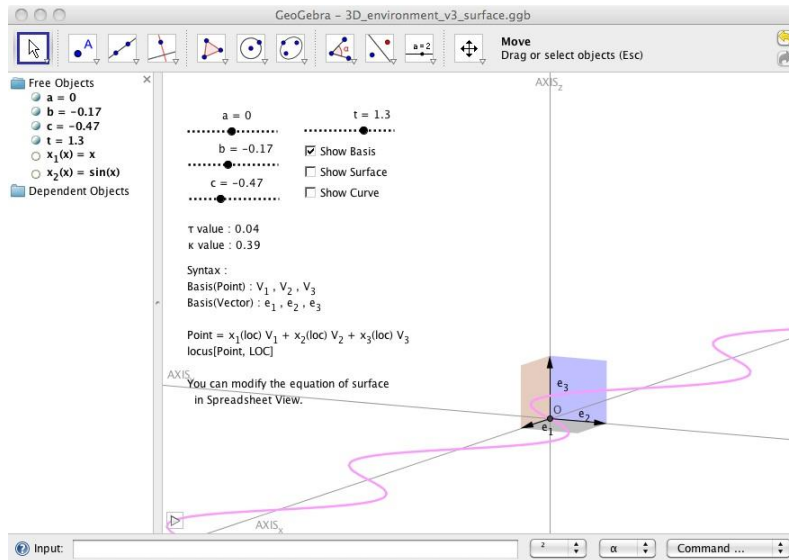

Figure 15: locus for representing domain curve, $(x, \ x^2, \ 0)$

4.2.1. Sphere

The function of sphere (radius 3) is $z = \pm\sqrt{9 - x^2 - y^2}$. Thus we change
`x_3(x) = sqrt(9 - x_1(x)^2 - x_2(x)^2 )`
 in algebra view and enter the equation to the cells from C22 to C47 using auto completion functionality in spreadsheet view (Figure 16).

We can change domain curve as new functions. For example, if the functions are
`x_1(t) = x` and `x_2(x) = x^2`,
we can get the graph of Figure 17.

4.2.2. Surface of Saddle

For representing surface of saddle, the equation is $z = -x^2 + y^2$.
Thus we define `x_3(x)` as $-x\_1(x)^2 + x\_2(x)^2$
and change the equation in spreadsheet view (Figure 18).

4.2.3. Torus

For representing torus, the equation is $z = \pm\sqrt{1 - (\sqrt{x^2 + y^2} - 2)^2}$. Thus we define

x_3(x) as `sqrt(1 - (sqrt(x_1(x)^2 + x_2(x)^2)-2)^2)`
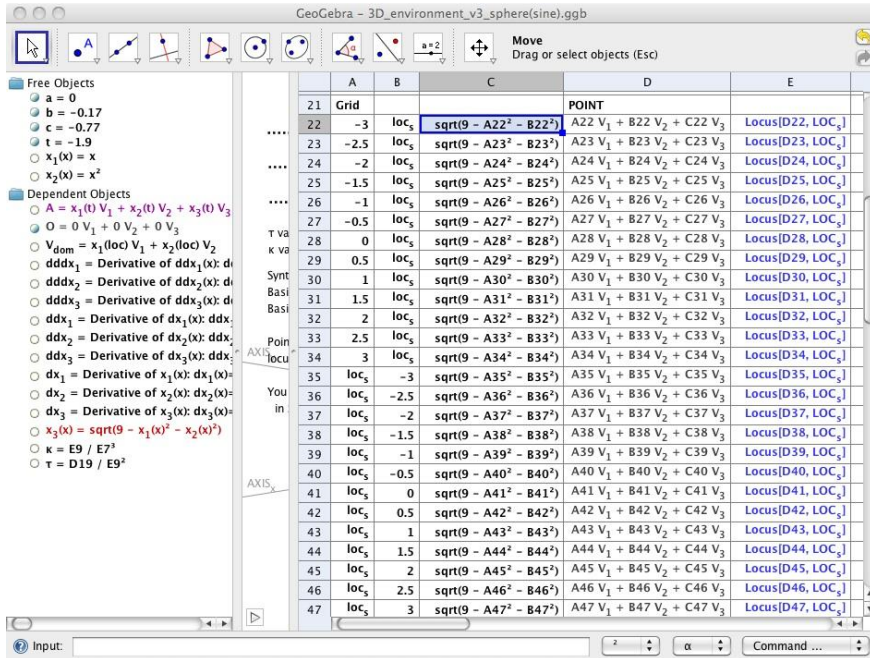 and change the equation in spreadsheet view (Figure 19).

Figure 16: Defining sphere and curve on sphere ($z = \sqrt{9 - x^2 - y^2}$)
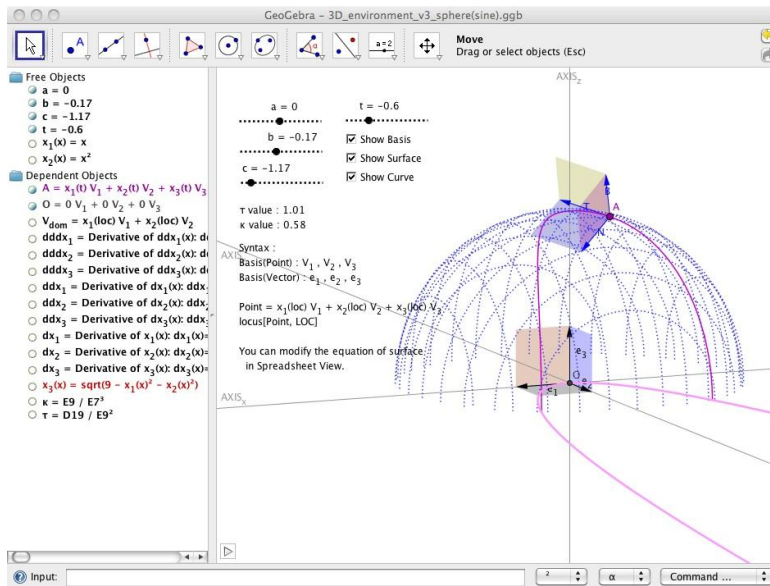
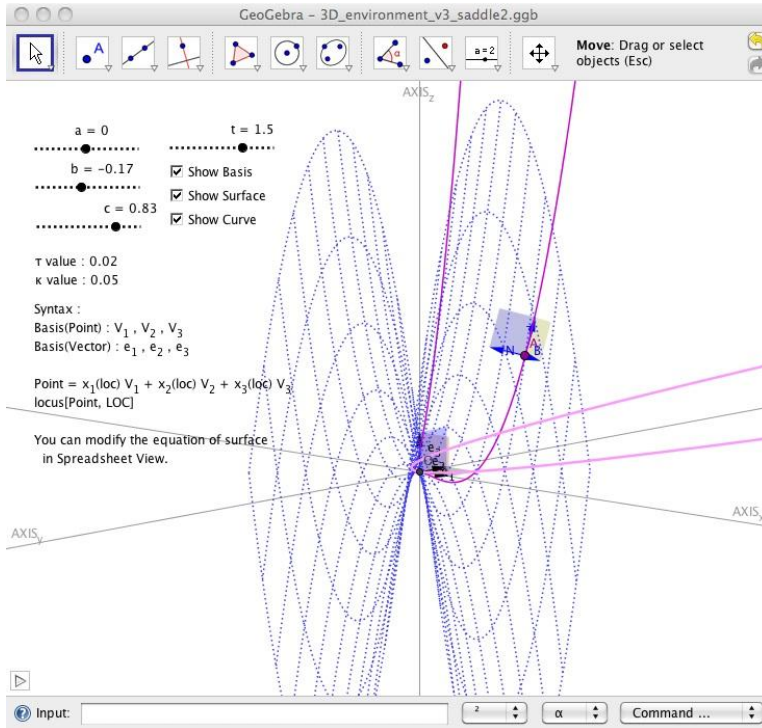**Figure 17:** Graph of curve on sphere (Domain: $y = x^2$ in *xy*-plain)

Figure 18: Graph of curve on surface of saddle (Domain: $y = x^2$ in xy-plain)
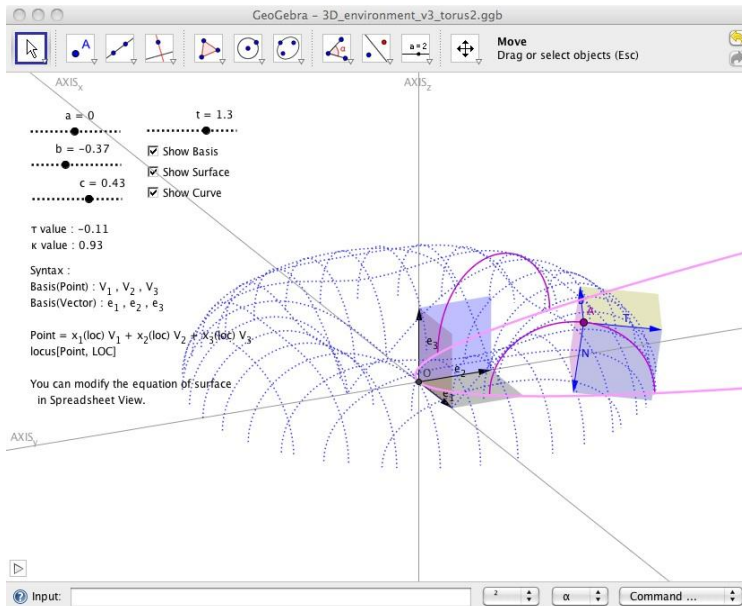

Figure 19: Graph of curve on torus (Domain: $y = x^2$ in xy-plain)

5. Conclusion

This study is an application for the use of Dynamic Mathematics Software, especially GeoGebra, as mathematics exploration environment. We explored 3-dimensional space with curves and surfaces using GeoGebra. For exploration, we constructed the environment for representing 3-dimensional space in GeoGebra according to Park *et al*.(2010)'s result. After that, we constructed Frenet-Serret frame of a curve in 3-dimensional space for observing movement of the variation vectors, T, N and B. In the process of constructing, we used the command for derivative function, derivative[], and the functionalities of spreadsheet (auto completion functionality, relative reference functionality). We observed many curves and movement of a point on a curve and the variation vectors, T, N and B. We also observed curves on many surfaces (sphere, surface of saddle and torus).

Bibliography

[Park10]   **J. Park, Y. Son, O. Kwon, H. Yang & K. Choi** – *Constructing 3D graph of function with GeoGebra(2D)*,Paper presented at First Eurasia Meeting of GeoGebra, Istanbul, Turkey, 2010.

[Wiki10]   **Wikipedia** – *Frenet-Serret formulas*, http://en.wikipedia.org/wiki/Frenet-Serret_formulas, 2010.