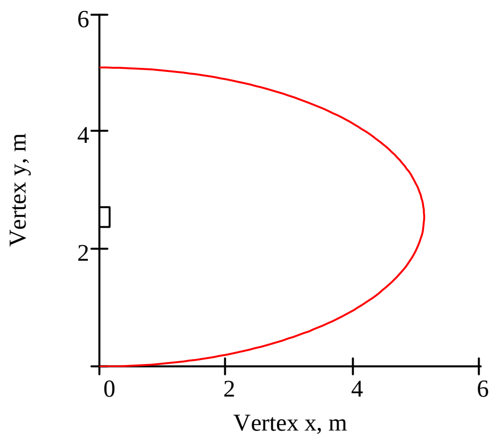
PROJECTILE MOTION *Ellipse of maxima*

If we examine the positions of the vertices (maxima) of a series of trajectories, all launched from a zero initial height, with the same initial velocity, an interesting pattern emerges. Consider the expressions for the vertex coordinates:

$$x_V(\theta) := \frac{v_0^2}{g}\sin(\theta)\cos(\theta) \qquad y_V(\theta) := \frac{v_0^2}{2\,g}\sin(\theta)^2$$



These parametric equations appear to be defining an ellipse, when θ ranges from zero to 90 degrees. We can convert these expressions to a single y(x) function and examine that result to see if it is in fact an ellipse. First we use some trig identities to write

$$x_V(\theta) = \frac{v_0^2}{2\,g}\sin(2\,\theta) \qquad y_V(\theta) = \frac{v_0^2}{4\,g}(1-\cos(2\,\theta)) \tag{1}$$

Using the parametric format first, we have from analytic geometry that an ellipse is defined by

$$x(\tau) = a\cos(\tau) \qquad y(\tau) = b\sin(\tau)$$

If we define $a := \dfrac{v_0^2}{2\,g}$ and $\tau = 2\,\theta$ then $x_V(\tau) = a\sin(\tau)$ $\quad y_V(\tau) = \dfrac{-a}{2}\cos(\tau) + \dfrac{a}{2}$

But it is also the case that

$$x_V(\tau) = a\sin(\tau) = a\cos\left(\frac{\pi}{2}-\tau\right) \qquad y_V(\tau) = \frac{a}{2} - \frac{a}{2}\sin\left(\frac{\pi}{2}-\tau\right)$$

and further, reversing the sign of the arguments (which must be the same) to get rid of the minus in y,

$$x_V(\tau) = a\cos\left(\tau-\frac{\pi}{2}\right) \qquad y_V(\tau) = \frac{a}{2} + \frac{a}{2}\sin\left(\tau-\frac{\pi}{2}\right)$$

since sin(-z) = -sin(z) and cos(-z) = cos(z). Now we see that the argument is just some new parameter ϕ, so that we have the ellipse format, with center at (0, a / 2), and minor axis in the y-direction:

$$x_V(\phi) = a\cos(\phi) \qquad y_V(\phi) = \frac{a}{2} + \frac{a}{2}\sin(\phi)$$

The position of the center of the ellipse is indicated by a square in the figure above. Note that ϕ is measured from this center, and sweeps from negative 90 degrees to positive 90 degrees.

For the cartesian form, we return to Eq(1), solve for the trig functions and then square:

$$\sin(2\,\theta)^2 = \frac{4\,x_V^2\,g^2}{v_0^4} \qquad \cos(2\,\theta)^2 = \left(1 - \frac{4\,g\,y_V}{v_0^2}\right)^2$$

Since $\sin(2\,\theta)^2 + \cos(2\,\theta)^2 = 1$ then $\dfrac{4\,x_V^2\,g^2}{v_0^4} + \left(1 - \dfrac{4\,g\,y_V}{v_0^2}\right)^2 = 1$

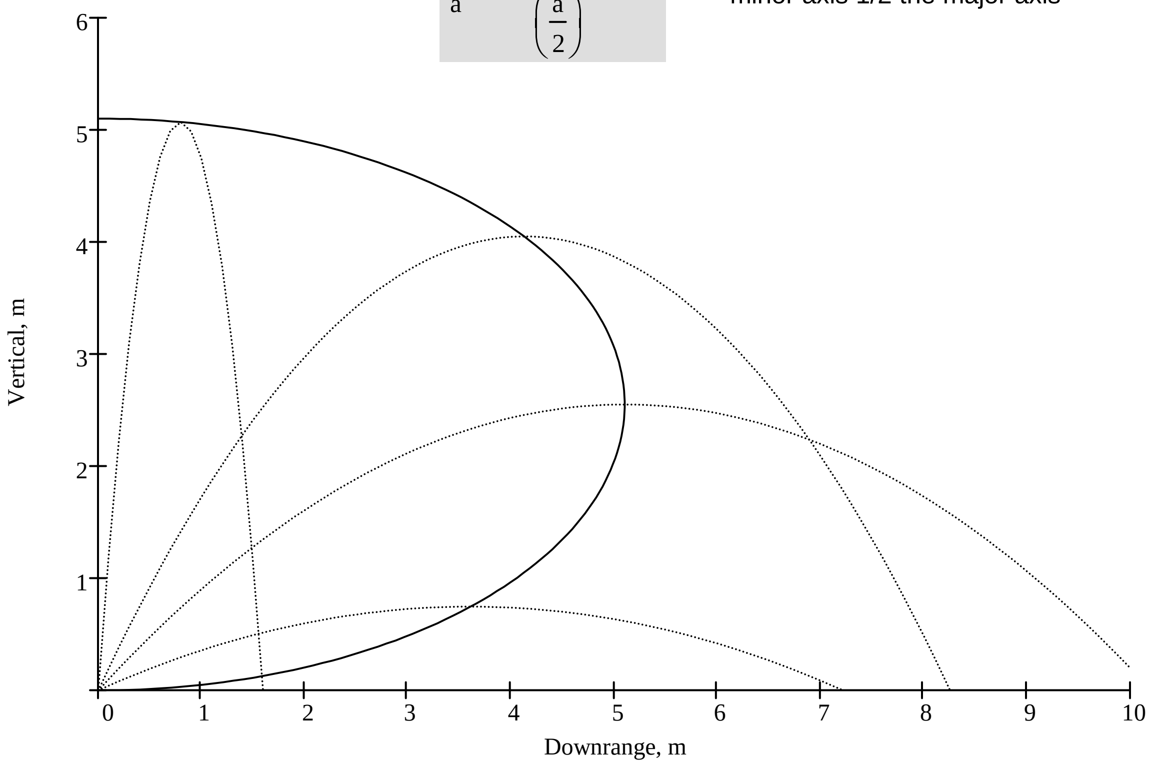
$$4\,x_V^2\,g^2 + \left(v_0^2 - 4\,g\,y_V\right)^2 = v_0^4 \qquad x_V^2 + \left(0 - 2\,v_0^2\,\frac{y_V}{g} + 4\,y_V^2\right) = 0 \qquad 4\,y^2 - 2\,\frac{v_0^2}{g}\,y + x^2$$

$$y^2 - \frac{v_0^2}{2\,g}\,y + \frac{v_0^4}{16\,g^2} - \frac{v_0^4}{16\,g^2} + \frac{x^2}{4} = 0 \qquad \left(y - \frac{v_0^2}{4\,g}\right)^2 + \frac{x^2}{4} = \frac{v_0^4}{16\,g^2} \qquad \boxed{x^2 + 4\left(y - \frac{v_0^2}{4\,g}\right)^2 = \frac{v_0^4}{4\,g^2}}$$

complete square

as given in Thomas p876

Using the definition of a, above, then $\boxed{\dfrac{x^2}{a^2} + \dfrac{\left(y - \frac{a}{2}\right)^2}{\left(\frac{a}{2}\right)^2} = 1}$ which is, again, the equation of an ellipse, with center at (0,α/2) and minor axis 1/2 the major axis



This figure shows a few example trajectories along with the ellipse defined above. Clearly the vertices fall along this ellipse.