**[MAA 5. 2] DERIVATIVES – BASIC RULES**

**SOLUTIONS**

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## O. Practice questions

**1.**

| $f(x)$ | $f'(x)$ |
|---|---|
| $f(x)=2x^5$ | $f'(x)=10x^4$ |
| $f(x)=\frac{2}{x^5}$ | $f'(x)=-10x^{-6}=-\frac{10}{x^6}$ |
| $f(x)=x^3+\ln x$ | $f'(x)=3x^2+\frac{1}{x}$ |
| $f(x)=2\sin x+3\cos x+5e^x$ | $f'(x)=2\cos x-3\sin x+5e^x$ |
| $f(x)=5x^3+2x^2+3x+7$ | $f'(x)=15x^2+4x+3$ |
| $f(x)=\frac{5}{x^3}+\frac{2}{x^2}+\frac{3}{x}+7$ | $f'(x)=-15x^{-4}-4x^{-3}-3x^{-2}=-\frac{14}{x^4}-\frac{4}{x^3}-\frac{3}{x^2}$ |
| $f(x)=\sqrt{x}-x+1$ | $f'(x)=\frac{1}{2\sqrt{x}}-1$ |
| $f(x)=6\sqrt[3]{x^5}$ | $f'(x)=6\times\frac{5}{3}x^{\frac{2}{3}}=10\sqrt[3]{x^2}$ |
| $f(x)=mx+c$ | $f'(x)=m$ |
| $f(x)=ax^2+bx+c$ | $f'(x)=2ax+b$ |

**2.**

| $f(x)$ | Simplify $f(x)$ | $f'(x)$ |
|---|---|---|
| $f(x)=x^2(2x+3)$ | $=2x^3+3x^2$ | $f'(x)=6x^2+6x$ |
| $f(x)=(3x+2)(2x+3)$ | $=6x^2+13x+6$ | $f'(x)=12x+13$ |
| $f(x)=2x^3+\frac{5}{x^3}+1$ | $=2x^3+5x^{-3}+1$ | $f'(x)=6x^2-15x^{-4}=6x^2-\frac{15}{x^4}$ |
| $f(x)=1+\frac{2}{x}+\frac{3}{x^2}$ | $=1+2x^{-1}+3x^{-2}$ | $f'(x)=-2x^{-2}-6x^{-3}=-\frac{2}{x^2}-\frac{6}{x^3}$ |
| $f(x)=x^2(1+\frac{2}{x}+\frac{3}{x^2})$ | $=x^2+2x+3$ | $f'(x)=2x+2$ |
| $f(x)=\frac{1+x+x^2}{x^2}$ | $=x^{-2}+x^{-1}+1$ | $f'(x)=-2x^{-3}-x^{-2}=-\frac{2}{x^3}-\frac{1}{x^2}$ |
| $f(x)=\frac{3x^5}{2}+\frac{2}{3x^4}$ | $=\frac{3}{2}x^5+\frac{2}{3}x^{-4}$ | $f'(x)=\frac{15}{2}x^4-\frac{8}{3}x^{-5}=\frac{15}{2}x^4-\frac{8}{3x^5}$ |
| $f(x)=\frac{2x^5+5x^2+1}{x^2}$ | $=2x^3+5+x^{-2}$ | $f'(x)=6x^2-2x^{-3}=6x^2-\frac{2}{x^3}$ |
| $f(x)=\frac{2x^5+5x^2+1}{3x^2}$ | $=\frac{2}{3}x^3+\frac{5}{3}+\frac{1}{3}x^{-2}$ | $f'(x)=2x^2-\frac{2}{3}x^{-3}=2x^2-\frac{2}{3x^3}$ |
| $f(x)=3x(\sqrt{x}+1)$ | $=3x^{\frac{3}{2}}+3x$ (or $3x^{1.5}+3x$) | $f'(x)=\frac{9}{2}x^{\frac{1}{2}}+3$ (or $4.5x^{0.5}+3$) |
| $f(x)=\sqrt{x}(2x+3\sqrt{x})$ | $=2x^{\frac{3}{2}}+3x$ (or $2x^{1.5}+3x$) | $=3x^{\frac{1}{2}}+3$ (or $3x^{0.5}+3$) |
| $f(x)=\frac{2x+3\sqrt{x}}{\sqrt{x}}$ | $=3x^{\frac{1}{2}}+3$ (or $3x^{0.5}+3$) | $=\frac{3}{2}x^{-\frac{1}{2}}=\frac{3}{2\sqrt{x}}$ (or $1.5x^{-0.5}=\frac{1.5}{\sqrt{x}}$) |

**3.**

| $y = f(x)$ | $\dfrac{dy}{dx}$ |
|---|---|
| $y = e^x \sin x$ | $y' = e^x \sin x + e^x \cos x$ |
| $y = e^x \cos x$ | $y' = e^x \cos x - e^x \sin x$ |
| $y = x^3 e^x$ | $y' = 3x^2 e^x + x^3 e^x$ |
| $y = x^2 \ln x$ | $y' = 2x \ln x + x^2 \dfrac{1}{x} = 2x \ln x + x$ |
| $y = \sqrt{x} \sin x$ | $y' = \dfrac{1}{2\sqrt{x}} \sin x + \sqrt{x} \cos x$ |
| $y = (2x+3)\cos x$ | $y' = 2\cos x - (2x+3)\sin x$ |
| $y = \dfrac{e^x}{\sin x}$ | $y' = \dfrac{e^x \sin x - e^x \cos x}{\sin^2 x}$ |
| $y = \dfrac{\sin x}{e^x}$ | $y' = \dfrac{e^x \cos x - e^x \sin x}{e^{2x}}$ |
| $y = \dfrac{e^x + 1}{\cos x}$ | $y' = \dfrac{e^x \cos x + (e^x + 1)\sin x}{\cos^2 x}$ |
| $y = \dfrac{2x-1}{3x+5}$ | $y' = \dfrac{2(3x+5) - 3(2x-1)}{(3x+5)^2} = \dfrac{13}{(3x+5)^2}$ |
| $y = \dfrac{2x+3}{\cos x}$ | $y' = \dfrac{2\cos x - (2x+3)\sin x}{\cos^2 x}$ |
| $y = \dfrac{\cos x}{2x+3}$ | $y' = \dfrac{-(2x+3)\cos x + 2\sin x}{(2x+3)^2}$ |
| $y = \dfrac{7x^3}{5} + \dfrac{7}{5x^3} + \dfrac{4x}{3} - \dfrac{4}{3x}$ | $y' = \dfrac{21}{5}x^2 + \dfrac{21}{5}x^{-4} + \dfrac{4}{3} - \dfrac{4}{3}x^{-2} \quad \left( = \dfrac{21x^2}{5} + \dfrac{21}{5x^4} + \dfrac{4}{3} - \dfrac{4}{3x^2} \right)$ |
| $y = x^2 + \ln x + x^2 \ln x$ | $y' = 2x + \dfrac{1}{x} + 2x \ln x + x^2 \dfrac{1}{x} = 3x + \dfrac{1}{x} + 2x \ln x$ |

**4.** (a) (i) $f'(x) = 6x^2 + \dfrac{1}{x}$ (ii) $f''(x) = 12x - \dfrac{1}{x^2}$

(b) (i) $f'(1) = 6 + 1 = 7$ (ii) $f''(x) = 12 - 1 = 11$

**5.** (a) $f'(x) = \dfrac{2x \sin x - x^2 \cos x}{\sin^2 x}$

(b) Directly by GDC (i) $f'\left(\dfrac{\pi}{2}\right) \cong 3.14$ (ii) $f'(1) \cong 1.61$

[Notice: the exact value for (i) is $f'\left(\dfrac{\pi}{2}\right) = \dfrac{\pi \sin\dfrac{\pi}{2} - \left(\dfrac{\pi}{2}\right)^2 \cos\dfrac{\pi}{2}}{\sin^2 \dfrac{\pi}{2}} = \dfrac{\pi - 0}{1} = \pi$ ]

**6.** (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = 2f'(x) + 3g'(x)$. At $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 23$

(b) $\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$. At $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 22$

(c) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$. At $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{9}$

(d) $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 5f'(x)$. At $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 26$

**7.** (a)

| $x$ | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| $f(x)$ | 2 | 3 | 4 | 6 |

(b)

| $x$ | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| $f'(x)$ | 0 | 1 | 2 | -2 |

(c)

| $x$ | 1.7 | 4.1 | 5.8 | 6.5 |
|---|---|---|---|---|
| $f'(x)$ | 0 | 2 | 2 | -2 |

(d) $x = 5,\ x = 7$.

**8.** $f'(x) = 10x - 3$

Gradient = 7 $\Rightarrow f'(x) = 7 \Leftrightarrow 10x - 3 = 7 \Leftrightarrow 10x = 10 \Leftrightarrow x = 1$

Then $y = f(1) = 2$

The coordinates are (1,2)

**9.** (a) $f'(3) = 0$

(b) $f(4) = 5$ (by symmetry)

(c) $f'(4) = -2$ (by symmetry)

## A. Exam style questions (SHORT)

**10.** (a) $f'(x) = 3x^2 - 4x - 0 = 3x^2 - 4x$

(b) Gradient at $x = 2$ : $f'(2) = 3 \times 4 - 4 \times 2 = 4$

**11.** $f(x) = 6x^{\frac{2}{3}}$

$$f'(x) = 4x^{-\frac{1}{3}} \left( = \frac{4}{x^{\frac{1}{3}}} = \frac{4}{\sqrt[3]{x}} \right)$$

**12.** $$h'(x) = \frac{(\cos x)(6) - (6x)(-\sin x)}{(\cos x)^2}$$

$$h'(0) = \frac{(\cos 0)(6) - (6 \times 0)(-\sin 0)}{(\cos 0)^2} = 6$$

**13.** (a) $g'(x) = 2\sin x + 2x\cos x$

(b) $g'(\pi) = 2\sin\pi + 2\pi\cos\pi = -2\pi$

**14.** (a) $x = \frac{1}{5}$

(b) $f'(x) = \frac{(5x-1)(6x)-(3x^2)(5)}{(5x-1)^2} = \frac{30x^2-6x-15x^2}{(5x-1)^2} = \frac{15x^2-6x}{(5x-1)^2}$

**15.** $f(1) = 1^2 - 3b + c + 2 = 0$

$f'(x) = 2x - 3\text{b}$,

$f'(3) = 6 - 3b = 0 \Rightarrow 3b = 6 \Rightarrow b = 2$

$1 - 3(2) + c + 2 = 0 \Rightarrow c = 3$

**16.** (a) $f'(x) = \ln x + x \cdot \frac{1}{x} - 1 = \ln x$

(b) $f'(x) = 1 \Leftrightarrow \ln x = 1 \Leftrightarrow x = \text{e}$

Then $y = f(\text{e}) = \text{e} \ln \text{e} - \text{e} = 0$, thus ( e ,0)

**17.** $f'(x) = 2\text{e}^x - 4 = 0 \Leftrightarrow \text{e}^x = 2 \Leftrightarrow x = \ln 2$

**18.** $f(x) = \tan x = \frac{\sin x}{\cos x}$

$f'(x) = \frac{\sin x \sin x - \cos x(-\cos x)}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

$f'(0) = 1$, $f'\left(\frac{\pi}{4}\right) = 2$, $f'\left(\frac{\pi}{3}\right) = 4$

They are in geometric sequence since $\frac{2}{1} = \frac{4}{2}$ $(= 2)$

**19.** $f'(x) = 3kx^2 - 30$

$f'(2) = 6 \Leftrightarrow 12k - 30 = 6 \Leftrightarrow 12k = 36 \Leftrightarrow k = 3$

**20.** (a) $f'(x) = k \cos x + 3$

(b) $k \cos\left(\frac{\pi}{3}\right) + 3 = 8$ □ $k\left(\frac{1}{2}\right) + 3 = 8$ □ $k = 10$

**21.** (a) (i) 1 (ii) 0.5

(b) (i) 0 (ii) $-\frac{1}{2}$

**22.** (a) (i) 1

(ii) 2

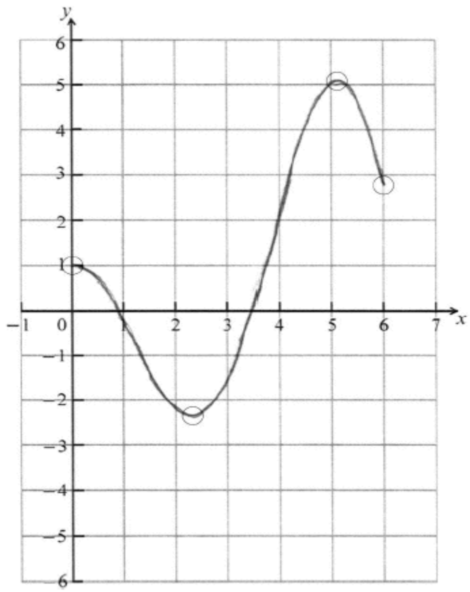
(iii) $f'(14) = f'(2)\left(\text{or } f'(5) \text{ or } f'(8)\right) = -1$

(b) There are five repeated periods of the graph, each with two solutions,

*ie* number of solutions is $5 \times 2 = 10$

**23.** (a) $f'(x) = \cos x - x\sin x$

(b)




***Mind:*** *correct domain, $0 \le x \le 6$ with endpoints in circles,*
*approximately correct shape,*
*local minimum and local maximum in circles.*

(c) $y \in [-2.38, 5.10]$

**24.** $f'(x) = 2ax + b$, $f''(x) = 2a$

$f(0) = 2 \Leftrightarrow c = 2$

$f'(0) = -3 \Leftrightarrow b = -3$

$f''(0) = 6 \Leftrightarrow 2a = 6 \Leftrightarrow a = 3$

**25.** $f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$f(0) = 2 = d$

$f'(1) = f(1) \to a + b + c + 2 = 3a + 2b + c$

$2 = 2a + b$

$f'(0) = -3 = c$

$f''(-1) = 6 = -6a + 2b$

$b = \frac{12}{5}, a = -\frac{1}{5}$

$f(x) = -\frac{1}{5}x^3 + \frac{12}{5}x^2 - 3x + 2$ (Accept $a = -\frac{1}{5}, b = \frac{12}{5}, c = -3, d = 2$)

## B. Exam style questions (LONG)

**26.** (a) $f(1)=5 \Leftrightarrow 2+a-5+b=5 \Leftrightarrow a+b=8$

(b) $f'(x)=6x^2+2ax-5$

(c) $f'(1)=7 \Leftrightarrow 6+2a-5=7 \Leftrightarrow 2a=6 \Leftrightarrow a=3$

$b=5$

(d) $f'(0)=-5$

(e) $f'(x)=7 \Leftrightarrow 6x^2+6x-5=7 \Leftrightarrow 6x^2+6x-12=0$

$\Leftrightarrow x^2+x-2=0$

Solutions: $x=1$ (expected, this is P) and $x=-2$

$y=2(-2)^3+3(-2)^2-5(-2)+5=-16+12+10+5=11$

Hence Q(-2, 11)

(f) $m_{PQ}=\frac{6}{-3}=-2$

$y-5=-2(x-1) \Leftrightarrow y=-2x+7$

**27.** (a) $f'(x)=4x^3$, $f''(x)=12x^2$, $f'''(x)=24x$, $f^{(4)}(x)=24$

(b) $g^{(4)}(x)=24$

(c) $h^{(m)}(x)=m!$

(d) $k(x)=\frac{1}{x}=x^{-1}$

(i) $k'(x)=-x^{-2}=-\frac{1}{x^2}$, $k''(x)=2x^{-3}=\frac{2}{x^3}$, $k'''(x)-6x^{-4}=-\frac{6}{x^4}$

$k^{(4)}(x)=24x^{-5}=\frac{24}{x^5}$

(ii) $k^{(n)}(x)=(-1)^n\frac{n!}{x^{n+1}}$