PROJECTILE MOTION *Galileo angles*

It was observed by Galileo that the same range can be obtained (ignoring air resistance) for two values of the initial angle. Let's see how this works. This is for the case of a zero initial height.

First, a simple way to show it. We have the zero initial height range equation

$$R = 2\,\frac{v_0^{\,2}}{g}\,\sin(\theta)\,\cos(\theta)$$

If we define

$$\phi = \frac{\pi}{2} - \theta$$

so that θ and ϕ are complementary angles, then it is the case that

$$\sin(\theta) = \cos(\phi) \qquad\qquad \cos(\theta) = \sin(\phi)$$

and then, since the order of multiplication is irrelevant

$$R = 2\,\frac{v_0^{\,2}}{g}\,\sin(\phi)\,\cos(\phi)$$

which produces the same R as before. Thus any two complementary angles $\theta_1 + \theta_2 = \frac{\pi}{2}$ will produce the same range. Galileo expressed this as a sum and difference of some quantity from 45 degrees, so that

$$\theta_1 = \frac{\pi}{4} + x \qquad \theta_2 = \frac{\pi}{4} - x \qquad \theta_1 + \theta_2 = \frac{\pi}{2}$$

But what is "x" here? This is a little more complicated. Re-write the range equation

$$R = \frac{v_0^{\,2}}{g}\,\sin(2\,\theta) \quad \text{and let} \quad \gamma = 2\,\theta \quad \text{so that} \quad \gamma = \operatorname{asin}\left(\frac{g\,R}{v_0^{\,2}}\right)$$

It can be shown that

$$\operatorname{asin}(z) + \operatorname{acos}(z) = \frac{\pi}{2} \qquad \text{so} \qquad \operatorname{asin}(z) = \frac{\pi}{2} - \operatorname{acos}(z)$$

Then $\gamma = \frac{\pi}{2} - \operatorname{acos}\left(\frac{g\,R}{v_0^{\,2}}\right)$ and $\theta = \frac{\pi}{4} - \frac{1}{2}\operatorname{acos}\left(\frac{g\,R}{v_0^{\,2}}\right)$ (This is half of it.) (1)

Next we use a trig factoid $\frac{g\,R}{v_0^{\,2}} = \sin(\gamma) = \sin(\pi - \gamma)$ so $\operatorname{asin}\left(\frac{g\,R}{v_0^{\,2}}\right) = \pi - \gamma$

and then $\gamma = \pi - \operatorname{asin}\left(\frac{g\,R}{v_0^{\,2}}\right)$ But this is the same as $\gamma = \pi - \left[\frac{\pi}{2} - \operatorname{acos}\left(\frac{g\,R}{v_0^{\,2}}\right)\right]$

which leads to

$$\theta = \frac{\pi}{4} + \frac{1}{2}\,\mathrm{acos}\left(\frac{g\,R}{v_0^{\,2}}\right) \qquad (2)$$

If we add Eq(1) and (2) we get 90 degrees, so the quantity "x" that we add or subtract from 45 degrees to get the same range is

$$\Delta\theta = \frac{1}{2}\,\mathrm{acos}\left(\frac{g\,R}{v_0^{\,2}}\right) \qquad (3)$$

So if we wanted to attain a specific range for a given initial velocity we can find the angle(s) this way, subject to the constraint

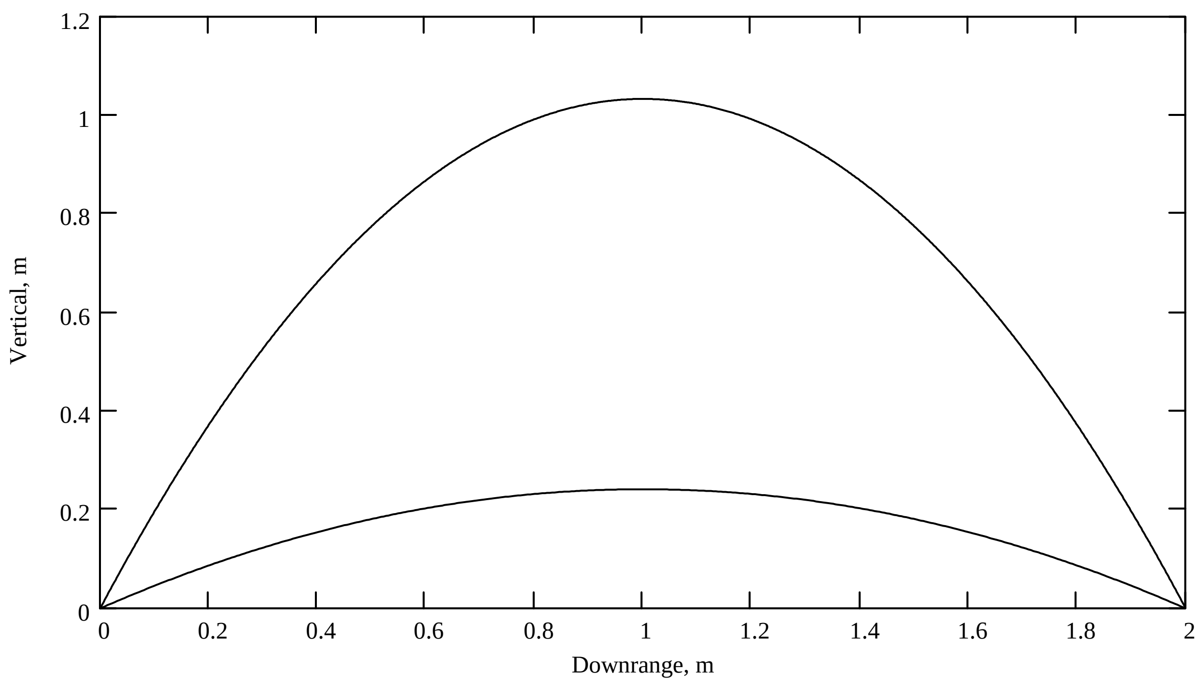
$$\frac{g\,R}{v_0^{\,2}} \le 1$$

which says that the maximum range attainable for a given initial velocity is

$$R_{max} = \frac{v_0^{\,2}}{g}$$

and the angle increment Eq(3) is zero, since cos(1) is zero, and the launch angle is thus 45 degrees.

Next we plot the trajectories for the Galileo angles. Note that the TOF will be different for the two angles, and that the initial y position is zero in both cases.



$\theta_1 \frac{180}{\pi} = 25.814$ TOF1 = 0.444 $\theta_2 \frac{180}{\pi} = 64.186$ TOF2 = 0.919

More examples of this phenomenon are shown in the targeting plots, which show two trajectories that can hit a given target, if the target is "reachable" at all. In the Targeting paper it is shown that there can be two trajectories, even from a nonzero initial height, and **to** a nonzero height, but the angles will not have the symmetry found by Galileo, which only happens for a zero initial height, and for a "target" which is also at ground level.