NEWTON'S BUCKET

William C. Evans Sept. 2019

Introduction

Newton proposed an experiment in the *Principia*, where a bucket is attached to rope, which is then twisted many times. Some water is placed in the bucket, and the bucket is released, so that the torque from the twisted rope causes it to begin spinning. At first the water does not rotate, while the bucket does, so that there is a relative rotational velocity between them. At this time the water surface is (essentially) flat. As the bucket spins, the water begins to spin, until its rotational angular velocity is equal to that of the bucket. At this point there is a zero rotational velocity between the water and the bucket. But now the water is seen to move away from the axis of rotation, and its surface takes on a curved shape.

To Newton, this movement of the water was proof that there is some absolute frame of reference, by which the water "knows" that it is rotating. Relative to something nearby, the bucket, it is not moving (rotating). This philosophical argument is beyond our scope here; there are many references about this controversy available. All we want to accomplish here is to describe the shape of the rotating water, regardless of how it knows that it is rotating.

The basic approach is to examine the forces acting on a small "chunk" of water at the surface. These forces will include gravity, a centrifugal force, and hydraulic forces that affect liquids. Using the "centrifugal" force is in itself debatable, since this is a fictitious force that applies only in a non-inertial (accelerating) frame of reference. So we are looking at the problem from the accelerated, non-inertial point of view of the *water*, rather than what appears to be the stationary (or at least non-accelerating) frame of reference of the surroundings where the bucket is located.

For further reading on this, there is a Wikipedia article on Newton's Bucket. A paper by Mungan and Lipscombe *"Newton's Rotating Water Bucket: A Simple Model"* Washington Academy of Sciences, Summer 2013, can be found online as a free PDF; this paper was most useful in developing the material presented below. The physics text *Elements of Newtonian Mechanics*, Knudsen and Hjorth, 3rd Ed., Springer 2002, pp.142-145 has a discussion of the bucket problem.

Water-Shape Model

As shown in the references, it is relatively straightforward to resolve the vector forces acting on the water surface, leading to the ordinary differential equation (ODE)

$$\frac{dz}{dr} = \frac{\omega^2}{g} r$$

where z is the vertical height above the bottom of the bucket (where $z = 0$); r is the distance from the bucket's rotation axis; ω is the angular velocity; g is the gravitational acceleration. The solution to this ODE is easily found to be

$$z(r) = z_0 + \frac{\omega^2}{2g} r^2$$

This is the equation of a parabola, which, when rotated about the central axis of the bucket, creates a paraboloidal volume. However, this solution, which appears at Wikipedia and in various textbooks, is incomplete, because the location of the vertex z_0 is left undefined (the Mungan and Lipscombe paper *does* address this).

Let us observe what happens when the angular velocity is zero, when the bucket and water are at rest. Then, for all r, $z(r)$ is just a constant γ, which represents the initial water height. Actually, r is constrained to be between zero and the radius of the bucket, R. (We will also, subsequently, have use for the height H of the bucket.) Then the initial volume of water in the bucket is just the volume of a cylinder

$$V_{initial} = \pi R^2 \gamma$$

We next make the essential assumption that no water leaves the bucket; we will see below that this places an upper limit on the angular velocity. Then, by conservation of mass, and the fact that water is not compressible, we know that the volume of water at *any* angular velocity below the water-loss value must be equal to the starting volume. So, if the water shape is paraboloidal, we can integrate that shape to get its volume, and then set that volume equal to the initial volume. In this way we hope to eliminate the unknown z_0.

Using the shell method of finding volumes of figures of revolution (taking shells parallel to the axis of rotation), we have for the total water volume, found below its surface, which has the vertical shape *z(r),* down to the bottom of the bucket:

$$\pi R^2 \gamma \;=\; 2\pi \int_0^R r\, z(r)\, dr$$

or

$$\pi R^2 \gamma = 2\pi \int_0^R r\left[z_0 + \frac{\omega^2}{2g} r^2 \right] dr \;=\; 2\pi \left\{ \int_0^R r\, z_0\, dr \;+\; \frac{\omega^2}{2g} \int_0^R r^3\, dr \right\}$$

which leads to

$$\gamma \;=\; z_0 + \frac{\omega^2}{4g} R^2 \quad \Rightarrow \quad z_0 \;=\; \gamma - \frac{\omega^2}{4g} R^2$$

so that we have found the height of z_0 as a function of the basic parameters of the problem. Now the complete equation of the water shape is

$$z(r) \;=\; \gamma - \frac{1}{g}\left(\frac{\omega R}{2} \right)^2 \;+\; \frac{\omega^2}{2g} r^2 \tag{1}$$

Angular-Velocity Relations

For our conservation-of-mass approach to apply, we must not have any water escaping from the bucket; this means that the maximum *z(r)* must be (just less than) the bucket height *H*. For this situation we can write

$$H = \;\gamma - \frac{1}{g}\left(\frac{\omega R}{2} \right)^2 \;+\; \frac{\omega^2}{2g} r^2$$

which we can solve to find the maximum angular velocity that keeps the water in the bucket

$$\omega_{\text{max}} = \frac{2}{R} \sqrt{g\left(H - \gamma \right)} \tag{2}$$

However, the situation is a bit more complicated than this. It turns out to be important to find the angular velocity that brings the paraboloid vertex just to the bottom of the bucket. This means that $z_0 = 0$, so that

$$\omega_{\text{bottom}} = \frac{2}{R} \sqrt{g\,\gamma} \tag{3}$$

and any ω larger than this will take the vertex below the bucket bottom, so that some of the bucket is exposed (not covered by water). The modeling breaks down under this condition, and it can be shown that, using the paraboloid shape, the water volume is not conserved. This means that the parabola model no longer applies when the angular velocity is this large.

So, we must have some restriction on the bucket setup (its size, and the initial water volume). For the water parabola vertex to be just at the bucket bottom while *simultaneously* not having any water escape at the top of the bucket, the two angular velocities given above must be equal. This quickly leads to the condition that the initial water level must be at least half the bucket height *H*, to get the water to "climb" just up to the top, while at the same time not exposing the bucket bottom.

A quantity that might be of interest is the height to which the water rises at a given angular velocity. For this case, we find *z(R)*, which turns out to be

$$z(R) = \gamma \,+\, \frac{1}{g}\left(\frac{\omega R}{2} \right)^2$$

which of course is measured from $z=0$, the bottom of the bucket.

Conservation-of-Mass Checks

Interestingly, it happens that, as the angular velocity ω changes, the shape of the paraboloid *z(r)* will intersect the original, initial-volume height γ at the *same* two values of r. See Figure 1.

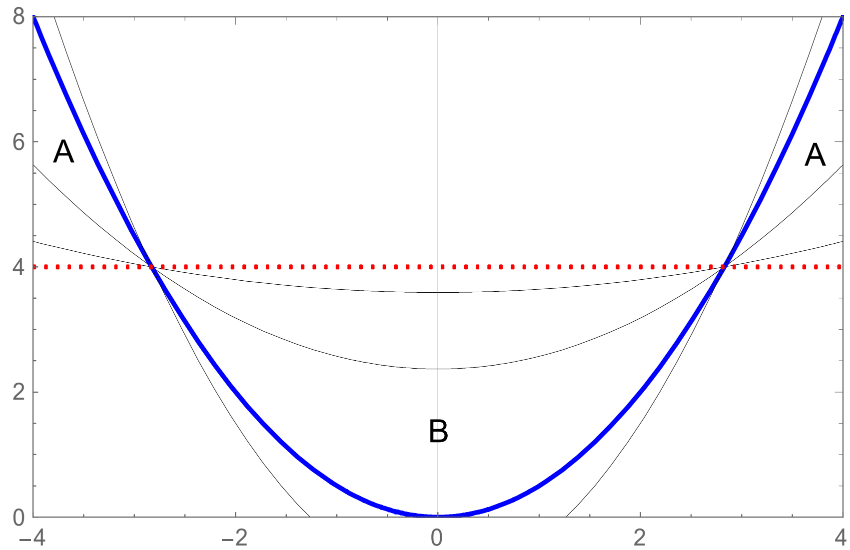

Figure 1. Example water surface shapes for several values of angular velocity. The dotted line is the initial water level. The largest velocity exposes the bucket bottom, so that the shape indicated is approximate. (This figure is a cross-section through a 3D volume.)

This implies that

$$\gamma \;=\; \gamma - \frac{1}{g}\left(\frac{\omega R}{2}\right)^2 \;+\; \frac{\omega^2}{2g} r^2$$

from which we find that, independent of ω, these values of r (actually, the radius of a circle, in 3D) will be

$$\pm \frac{\sqrt{2}}{2} R$$

These intersection points are of interest in exploring whether the volume of water *above* the γ-level (region A in the figure) and the volume "hollowed out" *below* that level by the water rotation (region B) are equal. By conservation of mass, they must be, but we would like to prove this. First, the volume of water B lost *below* the initial level; again use shell integration

$$V_{below} \;=\; 2\pi \int_0^{\frac{\sqrt{2}}{2}R} r\left[\,\gamma - z(r)\,\right] dr$$

which, using *Mathematica*, we find to be

$$V_{below} \;=\; \frac{\pi}{g}\left(\frac{\omega\, R^2}{4}\right)^2$$

and, in spite of appearances, this does have units of volume. Incidentally, this is just the volume of the (empty) paraboloid formed by rotating the half-region above the water but below γ about the central axis. This volume was of course found by Archimedes a long time ago, but we do the integration here, for practice, at least in setting it up- let *Mathematica* do the tedious work. Using Archimedes' proposition that the volume of the paraboloid is half that of an enclosing cylinder, we can get the result indicated above. (Note that when we did a shell integration above for a paraboloidal shape, leading to Eq. 1, that was for the water volume *below* the surface- here we want the paraboloidal *empty* volume *above* the water surface.)

Next, the volume A gained *above* the initial level, which takes the shape of a sort of triangular toroid in 3D as we rotate it about the central axis, is found using

$$V_{above} \;=\; 2\pi \int_{\frac{\sqrt{2}}{2}R}^{R} r\left[\,z(r) - \gamma\,\right] dr$$

which can be shown to produce the same result as just given. As expected, the water is conserved; so to speak, the hollowed-out water climbs up the sides of the bucket.