DYNAMIC VISUALIZATION as a principle for teaching and learning mathematics with digital mathematics learning environments

Hans-Jürgen Elschenbroich

Korschenbroich, Germany; elschenbroich@t-online.de

This paper presents the special role of dynamic visualization in teaching and learning mathematics in digital learning environments, illustrates it in examples, and situates it within a framework of instructional principles. The digital learning environments considered here consist of interactive dynamic GeoGebra worksheets. The focus lies on conceptual and subject-didactic reflections. No empirical study is presented; instead, the pedagogical experience gained from more than 3 decades of practice in school settings and teacher training in German-speaking countries is summarized and exemplarily condensed into four learning environments. This paper is addressed to teachers, teacher trainers and developers of digital learning environments and informs researchers in mathematics education.

Keywords: dynamic visualization, systematic variation, digital learning environment, instructional principles, teaching and learning.

Introduction

The increasing availability of dynamic mathematics software has transformed the possibilities for designing digital mathematics learning environments. Recent research has shown that teachers' decisions regarding the use of digital technology are not primarily driven by the availability of technological tools but by their perceived mathematical and pedagogical value (Gonscherowski & Rott, 2022). The educational potential of dynamic mathematics software cannot be understood solely in terms of its technical affordances. Rather, there is a need to articulate instructional principles that explain *why* and *how* technology can be used productively to support mathematical learning. This paper argues that dynamic visualization and systematic variation constitute two important principles for the design and use of interactive worksheets, currently known as digital mathematics learning environments (DMLE; Weinhandl, Baldinger & Kapplmüller, 2025). Rather than reporting an empirical study, the paper presents a conceptual and design-oriented analysis based on a collection of GeoGebra learning environments developed over 35 years. The Examples were selected because they represent typical mathematical domains and illustrate recurring design features that support mathematical exploration through dynamic visualization and systematic variation.

Conceptual Clarifications

The terms image/representation, illustration, and visualization are often conflated and thus used imprecisely. An image (*Abbild*) would, for example, be a photo that directly refers to an object. An illustration (*Veranschaulichung*) serves a didactic function: it is intended to ensure the target is (more) comprehensible. Visualizations (*Visualisierungen*) serve not only to illustrate acquired knowledge, they constitute an autonomous epistemic tool for knowledge generation. Static illustrations such as sketches, diagrams, and graphics have been common in the teaching and learning of mathematics for centuries. These are viewed and passively perceived. An initial form of dynamization emerged with

movable drawing instruments (pantograph, ellipse compass, integraph). Later, sequences of images and instructional films were launched as new dynamic media. In what follows, two relatively recent instructional principles are examined that are closely linked to the use of digital tools. Their integration within a canon of general instructional principles will be presented at the end of this article.

Dynamic Visualization

The term *dynamic visualization* emerged in the 1990s in the context of computer use in the United States, without being attributed to a specific individual. In German-speaking countries, it appeared around 1999 and became the topic of a plenary lecture at the annual conference of the GDM in 2004 (Elschenbroich, 2004). Dynamic mathematics software significantly expands the didactic possibilities of traditional media and enables the broad, straightforward access to dynamic visualization. Mathematical objects and configurations change over time, ideally not as passive consumption of videos, but through an *interactive* process involving learners (virtual) actions and the associated cognitive activities (Elschenbroich, 2004). Movable points and drag mode, sliders and traces/loci enrich mathematics instruction as new objects and become immediately accessible. But there is no didactic benefit at zero cost, because dynamic visualizations are intellectually demanding and are associated with increased complexity (Hölzl, 1994). In particular, prior knowledge plays a significant role in enabling not only to seeing something but truly comprehending it. Mere looking is not enough, as the old proverb says: "The eye sleeps until the mind awakens it with a question".

A common misconception is that the use of technology and multimedia learning environments by itself automatically triggers extensive learning processes (Schnotz & Lowe, 2003). Because dynamization entails the risk of cognitive overload and excessive acceleration, it is advisable to combine it with action-oriented phases of deceleration (Heintz et al., 2017; Mayer & Moreno, 2003) and to consolidate results in reflective and practice phases (Elschenbroich & Seebach, 2012–2014).

When working with DMLE, we can take this as a specific 'utilization scheme' in the sense of Rabardel (2002). Consequently, the dynamic learning environment cannot be analyzed and reduced to a mere 'artefact', but must be understood as an instrument, i.e. something used by a human in a defined, specific way, as an 'instrument' (Rabardel 2002).

Systematic Variation

Research on learners' use of Dynamic Geometry Software has shown that learners often engage in unsystematic ‘wild’ dragging. However, greater dynamism of this kind does not produce greater insight; rather, it may hinder it. What is crucial is “the conscious and goal-directed modification of ‘influencing factors’ as well as the observation and interpretation of the resulting changes in the dependent variables” (Roth, 2008; translation by the author). Scaffolding through appropriate task design, that is adapted to learners' prior knowledge, is therefore a central requirement for authors of dynamic learning environments. With suitable guidance by dynamic learning environments essential mathematical theorems at the secondary level can be discovered as invariants in drag mode and may even lead to visual proofs.

Digital Mathematics Learning Environments (DMLE)

In school, the general aim is teaching and learning, specifically in mathematics. As a methodological tool, digital mathematics learning environments in the form of DMLE have become established (Weinhandl et al, 2025). Previously referred to as *electronic* worksheets or *interactive* worksheets EIWOS (Elschenbroich & Seebach, 1999; Elschenbroich, 2001), these environments concretely implement instructional principles. They consist of an informative text, a dynamic file (today often created with GeoGebra), and learning-guiding tasks, possibly including solutions for self-study control. All components are integrated into a web-based file. They serve to strengthen self-directed activity and cognitive activation by giving learners the opportunity to "work on a topic using their individual learning and action capabilities" (Heitzer & Weigand, 2020; translation by the author). They also enable integrated assistance features that can be displayed (or hidden) with a mouse click or fingertip, and they can provide feedback and a solution for verification. In the age of AI this may imply a move forward in terms of quality. And it is important to emphasize that the use of DMLE must not be merely occasional, it must be supplemented by phases of practice and consolidation to ensure sustained learning success (Elschenbroich & Seebach, 2012–2014).

Examples of dynamic DMLE

All examples presented below are digital mathematics learning environments, compiled in a digital GeoGebra book (https://www.geogebra.org/m/m3cxvryy). They represent key areas of school mathematics and highlight the educational value of systematic variation and dynamic visualization.

The Pythagorean Theorem (Geometry)

The Pythagorean theorem is usually formulated for right-angled triangles. While mathematically correct, for discovery and understanding it is helpful to move out of this restriction and examine the relationship between a^2, b^2, and c^2 in arbitrary triangles. Although always possible in principle, this approach has been significantly promoted by dynamic mathematics software. A systematic variation ofthe vertex C, and thus of the angle γ, leads to three cases.

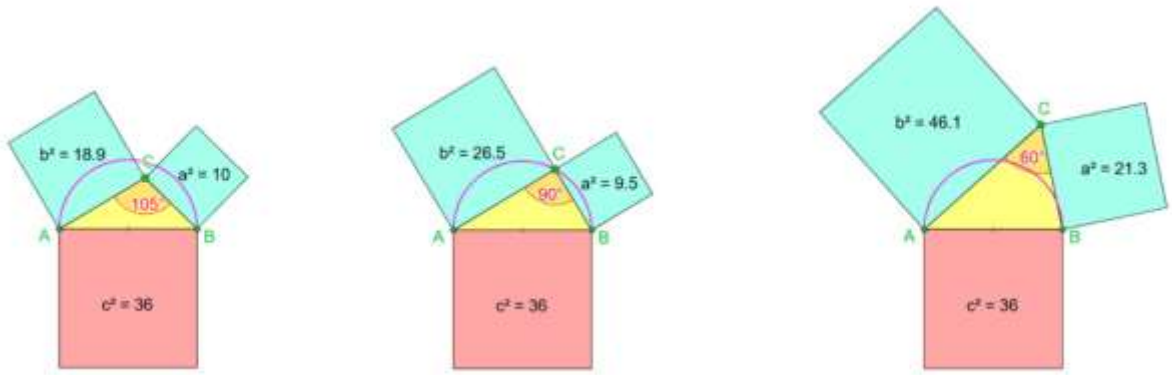


Figure 1a - c: Discovering the Pythagorean Theorem

In the right-angled case, this yields a visual proof of the Pythagorean theorem (and also of the Thales' theorem, since the Thales circle forms a boundary line). In the other two cases, it anticipates the cosine rule in qualitative form (Figure 1a - c).

Quadratic Functions and Equations (Algebra)

The investigation of quadratic functions and the solution of quadratic equations are standard topics in Mathematics 9th or 10th Grade in Germany. However, traditional algebraic methods such as completing the square and binomial formulas frequently cause difficulties. Conventional function plotters typically require starting from symbolic expressions and possibly varying parameters before examining graphs. Dynamic software allows direct manipulation of the graph itself (Figure 2a, b) like moving a *parabola template* and reverse the order: graphics first, then algebra.

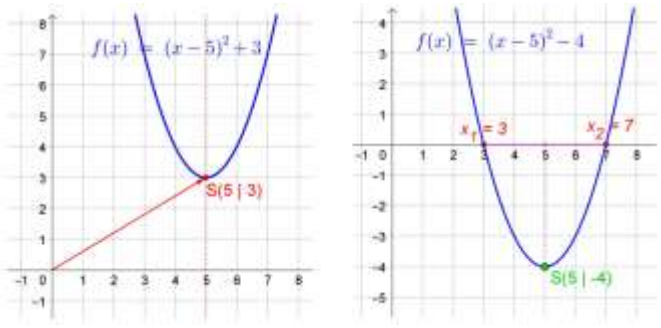


Figure 2a, b: Vertex form and zeros of quadratic functions

When object fixation is removed, one can drag the graph, changing its position and thus the function term and vertex while preserving its shape. Systematic variation is guided by specifying how and in which direction (x or y) dragging should initially occur, before allowing free movement. A particularly helpful feature is that arrow-key variation proceeds in steps of one unit, thus restricting attention to integer parameters. Learners can thereby discover that the quadratic function has the vertex form $f(x) = (x - x_S)^2 + y_S$ and that its zeros (if they exist) are $x_{1,2} = x_S \pm \sqrt{-y_S}$. This vertex-zero formula can afterwards be transformed into the standard *pq-formula* by using the fact that $x_S = -\frac{p}{2}$. Therefore you have direct a benefit by moving a parabola and you can see, what is happening on the algebraic interpretation of its position.

Slope, Local Linearity and Slope Curve (Calculus)

The *Function Loupe* builds upon A. Kirsch's graphical approach to the *Funktionenmikroskop* (function microscope) from the 1980s and dynamizes it. For a function f, a point A on the graph can be dragged. A number h, adjustable via a slider, defines a quadratic neighborhood of A, which is enlarged in a second window. As h decreases, a zoom effect occurs (Figure 3). For typical school functions, the graph segment eventually appears linear in the zoom window. This illustrates the basic concept of local linearity and allows specification of local slope at 7th Grade.

This initially purely graphical approach is linked to the standard school calculus by displaying the secants corresponding to A and h as well as the slope triangles. As h becomes smaller and smaller, the secants approach the tangent and converge. For very small h (minimum 0.0001), they are visually indistinguishable from each other in the global window and coincide locally with the function graph in the zoom window. This leads to the definition of tangent and differentiability.

Zooming can only be adequately achieved using digital tools. When enlarging a plotted graph step by step on a photocopier, the lines of the graph would become increasingly thicker. In this case, we have a form of virtual hands-on interaction through adjusting the slider h.

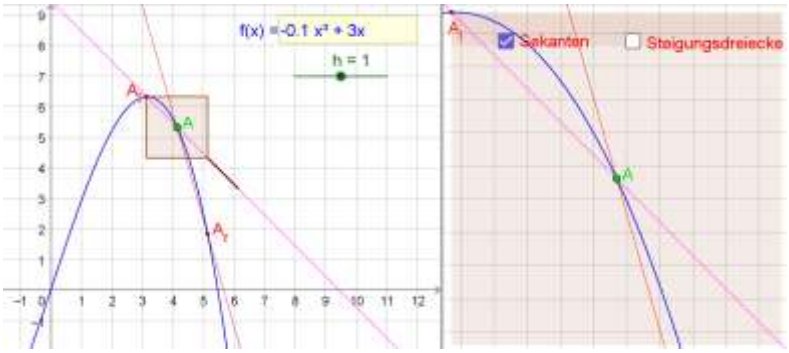


Figure 3: Zooming with the Function Loupe, starting with h = 1.

Here, we have two interconnected windows in which different aspects can be dynamically visualized in parallel. This is an exceptionally important and helpful feature of dynamic visualization (which, however, is available only in the *Classic* version of GeoGebra, NOT in the *Suite*).

If you can calculate or measure the slope m of the graph of a function f at *one* point A in a digital environment, you can do so at *any* points. This allows you to plot the graph of the derivative function f' as the locus curve of $Z = (a, m)$ without needing to know the term of the derivative function f'. This is a digital simulation of the mechanical device called the *Differentiograph* (Figure 4).

In this way, you gain a graphical, calculus-free access to the derivative function by first obtaining the slope curve and then considering what function formula might match it.

Archimedean Solids (3D Geometry)

If the vertices of Platonic solids are truncated uniformly, Archimedean solids may result, provided that the resulting faces are regular polygons. Dynamic visualization demonstrates its particular strength here.

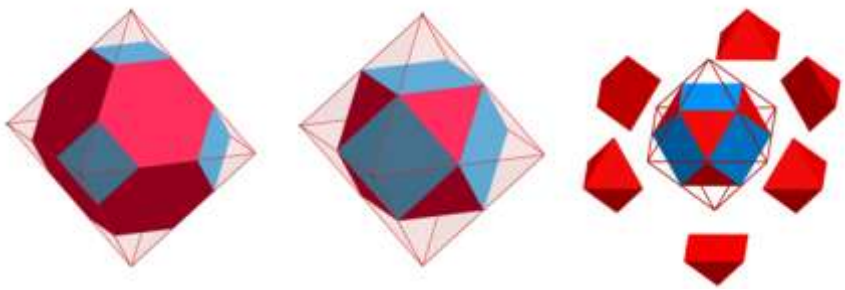


Figure 4a - c: Truncated Octahedron, Cuboctahedron and exploded view

With dynamic visualizations, you can explore uniform truncation of polyhedral using sliders and spheres with adjustable radius (Dutkowski, 2025). This allows interactive examination of changes, optionally displaying a translucent original body alongside the truncated solid (Figure 4a, b). With some effort, it is also possible to create a dynamic exploded view using sliders for a controlled and systematic blasting of well-defined components (Figure 4c).

The systematic variations of the radius r can only be carried out with the aid of digital mathematics tools, as can the parallel visualization of both the original object the octahedron and the resulting truncated solids. Working with physical models, on the other hand, is irreversible, difficult, and also problematic from a safety perspective. Implementing the construction in a dynamic 3D software environment, however, allows all changes to be viewed immediately on screen. Moreover, the object can be rotated, observed from different viewpoints, and the projection type (orthographic, oblique parallel, or central) can be changed.

It should also be noted that constructing such a model is quite time-consuming and can hardly be accomplished collectively in a classroom setting. Thus, teachers are needed as authors and experts to design and provide such learning environments.

Instructional Principles

Dynamic visualization and **systematic variation** should not be understood as isolated instructional principles. Dynamic visualization enriches the principle of multiple representations, while systematic variation provides a digital implementation of guided mathematical inquiry. Across the examples, recurring design features become visible. These include constrained interaction, dynamic visualization, systematic parameter variation, linked representations, parallel visualizations, and opportunities for reflection. Together, these features may be understood as design principles for digital mathematics learning environments (Elschenbroich & Dutkowski, 2023; Heintz et al, 2017). They complement established principles (Wittmann 1974/ 1981, Heitzer & Weigand, 2020;), that have been formulated by leading educators and developmental psychologists.

- **Genetic principle**
  Formulated by Martin Wagenschein, this principle advocates orientation toward the genesis of mathematics without retracing all historical detours (historical-genetic aspect). In the Merano Reform associated with Felix Klein, a demand was made to adapt more than before to the natural course of intellectual development (individual-genetic aspect).
- **Spiral principle**
  This was formulated by the educational psychologist Jerome A. Bruner, who advocated a spiral curriculum structured instead of disconnected subject areas, linking it with central ideas as guiding themes. According to Bruner, these ideas can be made accessible to every child at any age in an intellectually honest form appropriate to their level of development.
- **Operative principle**
  This is associated with the developmental psychologist Jean Piaget, who understood thinking as internalized or mental action. This idea is reflected today in the concept of action orientation, which uses concrete materials, drawings, and texts in classroom instruction.
- **Principle of multiple representations**
  Bruner proposed the idea “that one can know something in three ways: by doing it, by representing it through images, and by representing it through symbolic means such as language” (Bruner, 1966). In the English-speaking world this is also known as the CPA approach (**c**oncrete, **p**ictorial, **a**bstract); in German speaking contexts as the EIS principle

(**e**nactive, **i**conic, **s**ymbolic), later extended by Hole to the CEIS model to include the overarching aspect of computer use (Hole, 1988). The computer enables a stronger interconnection of Bruner's three modes of representation, which can be illustrated by a three-dimensional pyramid (Elschenbroich & Dutkowski, 2023).

Those instructional principles are general principles or rules of action for designing instruction that apply across all subjects. Observing them increases and ensures the efficiency and quality of teaching (Wiater, 2014). They are "principles for the design and evaluation of instruction which are based, on the one hand, on normative considerations and, on the other hand, on practical teaching experience and empirical findings. They integrate insights from psychological learning theories and present experiences from classroom practice in a condensed and abbreviated form." (Heitzer & Weigand, 2020; translation by the author).

Conclusion

This paper has argued that dynamic visualization and systematic variation can be regarded as special instructional principles for digital mathematics learning environments. They are embedded within a broader set of instructional principles, which gains specific significance in the design and use of digital learning environments. The computer connects the various levels and modes of learning and knowledge. Dynamic tools as drag mode, sliders, trace mode or loci, linked windows, and parallel representations play an important role in active and interactive learning.

In addition to the general didactic principles for classroom use, there are specific design principles for the development of DLME: integer step sizes for sliders where appropriate, very small values in sliders to visualize a limit process, reducing the burden of extensive calculations, parallel use of multiple windows, possibly with a magnification effect.

This perspective is consistent with findings by Gonscherowski and Rott (2022), who reported that both pre-service and in-service mathematics teachers justify the use of digital technology primarily when it provides added value to mathematical understanding, visualization, exploration, or learner activity. Their findings reinforce the argument that dynamic visualization should not be viewed as a technological feature alone but as a pedagogically meaningful design principle that requires careful orchestration.

References

Bruner, J. S. (1966): Introduction: Cognitive Growth and Education. In: Bruner, J. S.; Olver, R. R. & Greenfield, P. (1966): *Studies in Cognitive Growth*. John Wiley & Sons.

Dutkowski, W. (2025): Archimedische Körper – Faszination zur Regeometrisierung des Geometrie-unterrichts. In: *Beiträge zum Mathematikunterricht 2025, 361–364*, WTM-Verlag.

Elschenbroich, H.-J. & Dutkowski, W. (2023): „Erfolgreicher Mathematikunterricht" vor 25 Jahren und heute. Teil 1. In: *MNU Journal 5/2023*, 401–406. https://www.geogebra.org/m/hgp4fsp4#material/wbha7ujp

Elschenbroich, H.-J. & Seebach, G. (2012–2014): *Geometrie entdecken! Mit GeoGebra – Teil 1–3. MasterTool Themenpaket*. co.Tec, Rosenheim.

Elschenbroich, H.-J. & Seebach, G. (1999): *Dynamisch Geometrie entdecken. Elektronische Arbeitsblätter mit Euklid*. Verlag Dümmler-Stam.

Elschenbroich, H.-J. (2004): Dynamische Visualisierung durch neue Medien. In: *Beiträge zum Mathematikunterricht 2004*. Verlag Franzbecker, 2004, 7–14.

Elschenbroich, H.-J. (2001): *Electronic Interactive Worksheets: A visual and dynamic way to learn geometry*. Plenary talk, Cabri World 2001, Montreal.

Gonscherowski, P. & Rott, B. (2022). How Do Pre-/In-Service Mathematics Teachers Reason for or against the Use of Digital Technology in Teaching? Mathematics, 10(13), 2345. https://doi.org/10.3390/math10132345

Heintz, G., Elschenbroich, H.-J., Laakmann, H., Langlotz, H., Rüsing, M., Schacht, F., Schmidt, R. & Tietz, C. (2017): *Werkzeugkompetenzen. Kompetent mit digitalen Werkzeugen Mathematik betreiben*. MNU & T³. Verlag Medienstatt.

Heitzer, J. & Weigand, H.-G. (2020): Mathematikdidaktische Prinzipien. In: *mathematik lehren 223*, S. 2–5.

Hole, V. (1998): *Erfolgreicher Mathematikunterricht mit dem Computer. Methodische und didaktische Grundfragen in der Sekundarstufe I*. Auer.

Hölzl, R. (1994): *Im Zugmodus der Cabri-Geometrie: Interaktionsstudien und Analysen zum Mathematiklernen mit dem Computer*. Deutscher Studien Verlag.

Marton, F. & Tsui, A. B. M. (Eds.). (2004). *Classroom discourse and the space of learning*. Mahwah, NJ: Lawrence Erlbaum Associates.

Mayer, R. E. & Moreno, R. (2003). Nine ways to reduce cognitive load in multimedia learning. *Educational Psychologist, 38*(1), 43–52. https://doi.org/10.1207/S15326985EP3801_6

Rabardel, P. (2002). *People and technology: A cognitive approach to contemporary instruments*. Université Paris 8. https://hal.archives-ouvertes.fr/hal-01020705.

Roth, J. (2008): Systematische Variation. Eine Lernumgebung vernetzt Geometrie und Algebra. In: *mathematik lehren 146*, 17–21.

Schnotz, W. & Lowe, R. K. (2003). *External and internal representations in multimedia learning: Introduction*. *Learning and Instruction, 13*(2), 117–123. https://doi.org/10.1016/S0959-4752(02)00015-4.

Weinhandl, R., Baldinger, S. & Kapplmüller, M. (2025*).* Educational design characteristics of digital mathematics learning environments from a student perspective. In: *International Journal of Mathematical Education in Science and Technology*, 1–27. https://doi.org/10.1080/0020739X.2025.2562275.

Wiater, W. (2014). *Unterrichtsprinzipien* (6th ed.). Auer.

Wittmann, E. Ch. (1981). *Grundfragen des Mathematikunterrichts* (6th ed.; 1st ed. 1974). Vieweg.