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A Diophantine equation with exponential and power functions.

Consider the Diophantine equation given by

$(*)$: $a^x = x^q$. *a, x and q are natural numbers,* $a > 1, x > 1$.

Theorem A.

Assume a is of the form $a = s^v, s, v \in \mathbb{N}$, *with s prime and* $v = 1$ *or prime. Then*

The equation has a solution in $x \Leftrightarrow$ *(if and only if) q is of one of the* 2 *forms*

1. $q = vs^{s^r - r}, r \in \mathbb{N}_0$, *the solution is* $x = s^{s^r}$

2. $q = s^{vs^r - r}, r \in \mathbb{N}_0$, *the solution is* $x = s^{vs^r}$

Proof.

In the proof this basic result is used.

Let c be prime, a, b natural numbers. c is a divisor in ab $\Longrightarrow$ *c is a divisor in a or b*

Therefore x must a power of s.

Assume then that $x = s^p$. *The equation then becomes*

$$(s^v)^{s^p} = (s^p)^q \Leftrightarrow vs^p = pq \Leftrightarrow q = \frac{vs^p}{p}$$

Taking the options for p case by case for p leads to the result.

1. $p = s^r, r \in \mathbb{N}_0, q = \frac{vs^{s^r}}{s^r} = vs^{s^r - r}, x = s^p = s^{s^r}$

2. $p = vs^r, r \in \mathbb{N}_0, q = \frac{vs^{vs^r}}{vs^r} = s^{vs^r - r}, x = s^p = s^{vs^r}$ *.QED.*

Theorem B.

Assume a is of the form $a = s^{uv}, s, v \in \mathbb{N}$, *with s prime, v prime or* $v = 1$,

u *prime or* $u = 1$. *Then*

The equation has a solution in $x \Leftrightarrow$ *(if and only if) q is of one of the* 4 *forms*

1. $q = uvs^{s^r - r}, r \in \mathbb{N}_0, x = s^{s^r}$

2. $q = us^{vs^r - r}, r \in \mathbb{N}_0, x = s^{vs^r}$

3. $q = vs^{us^r - r}, r \in \mathbb{N}_0, x = s^{us^r}$

4. $q = s^{uvs^r - r}, r \in \mathbb{N}_0, x = s^{uvs^r}$

$Proof.$

$Assume\ then\ that\ x = s^p.\ The\ equation\ a^x = x^q\ then\ becomes$

$(s^{uv})^{s^p} = (s^p)^q \Leftrightarrow uvs^p = pq \Leftrightarrow q = \frac{uvs^p}{p}.Then\ case\ by\ case:$

1. $p = s^r, r \in \mathbb{N}_0, q = \frac{uvs^{s^r}}{s^r} = uvs^{s^r - r}, x = s^p = s^{s^r}$
2. $p = vs^r, r \in \mathbb{N}_0, q = \frac{uvs^{vs^r}}{vs^r} = us^{vs^r - r}, x = s^p = s^{vs^r}$
3. $p = us^r, r \in \mathbb{N}_0, q = \frac{uvs^{us^r}}{us^r} = vs^{us^r - r}, x = s^p = s^{us^r}$
4. $p = uvs^r, r \in \mathbb{N}_0, q = \frac{uvs^{uvs^r}}{uvs^r} = s^{uvs^r - r}, x = s^p = s^{uvs^r}$

Case $u = v$.

$The\ result\ then\ becomes$

$Assume\ a\ is\ of\ the\ form\ a = s^{v^2}, s, v \in \mathbb{N}, with\ s\ prime, v\ prime\ or\ v = 1. Then$

$The\ equation\ has\ a\ solution\ in\ x \Leftrightarrow (if\ and\ only\ if)\ q\ is\ of\ one\ of\ the\ 3\ forms$

1. $q = v^2 s^{s^r - r}, r \in \mathbb{N}_0, x = s^{s^r}$
2. $q = vs^{vs^r - r}, r \in \mathbb{N}_0, x = s^{vs^r}$
3. $q = s^{v^2 s^r - r}, r \in \mathbb{N}_0, x = s^{v^2 s^r}$

$Theorem\ C.$

$Assume\ a\ is\ of\ the\ form\ a = s^{v^n}, s, v, n \in \mathbb{N}, with\ s\ and\ v\ prime. Then$

$The\ equation\ has\ a\ solution\ in\ x \Leftrightarrow (if\ and\ only\ if)\ q\ is\ of\ one\ of\ the\ xx\ forms$

$Theorem\ D.$

$(*): a^x = x^q.\quad a, x\ and\ q\ are\ natural\ numbers, a > 1, x > 1.$

$Then\ a\ is\ a\ power\ of\ x\ or\ vice\ versa.$

$Proof. Prime\ factorization.$

$(*)$: $a^x = x^q$. a, x and q are natural numbers, $a > 1, x > 1$.

Assume a is of the form $a = s^v, s, v \in \mathbb{N}$, with s prime and $v = 1$ or prime. Then

The equation has a solution in $x \Leftrightarrow$ (if and only if) q is of one of the 2 forms

1. $q = vs^{s^r - r}, r \in \mathbb{N}_0$, *the solution is* $x = s^{s^r}$ 2. $q = s^{vs^r - r}, r \in \mathbb{N}_0$, *the solution is* $x = s^{vs^r}$

Let c be prime, a, b natural numbers. c is a divisor in $ab \Rightarrow c$ is a divisor in a or b

Assume then that $x = s^p$. The equation then becomes

$$(s^v)^{s^p} = (s^p)^q \Leftrightarrow vs^p = pq \Leftrightarrow q = \frac{vs^p}{p}$$

Taking the options for p case by case for p leads to the result.

1. $p = s^r, r \in \mathbb{N}_0, q = \frac{vs^{s^r}}{s^r} = vs^{s^r - r}, x = s^p = s^{s^r}$

2. $p = vs^r, r \in \mathbb{N}_0, q = \frac{vs^{vs^r}}{vs^r} = s^{vs^r - r}, x = s^p = s^{vs^r}$.

Theorem B.

Assume a is of the form $a = s^{uv}, s, v \in \mathbb{N}$, with s prime, v prime or $v = 1$, u prime or $u = 1$. Then

The equation has a solution in $x \Leftrightarrow$ (if and only if) q is of one of the 4 forms

1. $q = uvs^{s^r - r}, r \in \mathbb{N}_0, x = s^{s^r}$
2. $q = us^{vs^r - r}, r \in \mathbb{N}_0, x = s^{vs^r}$
3. $q = vs^{us^r - r}, r \in \mathbb{N}_0, x = s^{us^r}$
4. $q = s^{uvs^r - r}, r \in \mathbb{N}_0, x = s^{uvs^r}$

Proof.

Assume then that $x = s^p$. The equation then becomes

$$(s^{uv})^{s^p} = (s^p)^q \Leftrightarrow uvs^p = pq \Leftrightarrow q = \frac{uvs^p}{p}$$

1. $p = s^r, r \in \mathbb{N}_0, q = \frac{uvs^{s^r}}{s^r} = uvs^{s^r - r}, x = s^p = s^{s^r}$

2. $p = vs^r, r \in \mathbb{N}_0, q = \frac{uvs^{vs^r}}{vs^r} = us^{vs^r - r}, x = s^p = s^{vs^r}$

3. $p = us^r, r \in \mathbb{N}_0, q = \frac{uvs^{us^r}}{us^r} = vs^{us^r - r}, x = s^p = s^{us^r}$

4. $p = uvs^r, r \in \mathbb{N}_0, q = \frac{uvs^{uvs^r}}{uvs^r} = s^{uvs^r - r}, x = s^p = s^{uvs^r}$

QED.

Case $u = v$.

The result then becomes

$Assume\ a\ is\ of\ the\ form\ a = s^{v^2}, s, v \in \mathbb{N}, with\ s\ prime, v\ prime\ or\ v = 1. Then$

$The\ equation\ has\ a\ solution\ in\ x \Leftrightarrow (if\ and\ only\ if)\ q\ is\ of\ one\ of\ the\ 4\ forms$

1. $q = v^2 s^{s^r - r}, r \in \mathbb{N}_0, x = s^{s^r}$
2. $q = vs^{vs^r - r}, r \in \mathbb{N}_0, x = s^{vs^r}$
3. $q = s^{v^2 s^r - r}, r \in \mathbb{N}_0, x = s^{v^2 s^r}$

$Example.$

$Let\ s = 2, v = 1.\ a = s^v = 2^1 = 2, a^x = x^q\ becomes\ 2^x = x^q$

$1. q = s^v = 2^1 = 2, the\ solution\ is\ x = s^v = 2$

$2. q = vs^{s-1} = 1 \cdot 2^{2-1} = 2, the\ solution\ is\ x = s^s = 2^2 = 4$

$3. q = sv = 2 \cdot 1, the\ solution\ is\ x = s = 2$

$4. q = vs^{s^r - r} = 1 \cdot 2^{2^r - r} = 2^{2^r - r}, r \in \mathbb{N}, the\ solution\ is\ x = s^{s^r} = 2^{2^r}$

$5. q = s^{vs^r - r} = 2^{2^r - r}, r \in \mathbb{N}, the\ solution\ is\ x = s^{vs^r} = 2^{2^r}.$

Examples to number 5. *We have* $q = 2^{2^r - r}$, $x = 2^{2^r}$.

Let $r = 2$, *then* $q = 2^{2^r - r} = 2^{2^2 - 2} = 2^2 = 4, x = 2^{2^r} = 2^4 = 16,$

$2^x = x^q \rightarrow 2^x = x^4, with\ 2^{16} = 16^4$

Let $r = 3$, *then* $q = 2^{2^r - r} = 2^{2^3 - 3} = 2^5 = 32, x = 2^{2^r} = 2^{2^3} = 2^8 = 256,$

$2^x = x^q \rightarrow 2^x = x^{64}, with\ 2^{256} = 256^{32}$

Let $r = 4$, *then* $q = 2^{2^r - r} = 2^{2^4 - 4} = 2^{12} = 4096, x = 2^{2^r} = 2^{2^4} = 2^{16} = 65536,$

$2^x = x^q \rightarrow 2^x = x^{4096}, 2^{65536} = 65536^{4096} \Leftrightarrow 2^{2^{16}} = (2^{16})^{4096}, true$

Example.

Let $s = 7, v = 1.$ $a = s^v = 7^1 = 7, a^x = x^q$ *becomes* $7^x = x^q$

$1. q = s^v = 7^1 = 7, the\ solution\ is\ x = s^v = 7$

$2. q = vs^{s-1} = 1 \cdot 7^{7-1} = 7^6 = 117649\ , the\ solution\ is\ x = s^s = 7^7 = 823543$

$7^x = x^q \Leftrightarrow 7^x = x^{7^6} \Leftrightarrow 7^x = x^{117649}$

$3. q = sv = 7 \cdot 1, the\ solution\ is\ x = s = 7$

$4. q = vs^{s^r - r} = 1 \cdot 7^{7^r - r} = 7^{7^r - r}, r\ \in \mathbb{N}, the\ solution\ is\ x = s^{s^r} = 7^{7^r}$

Let $r = 2. q = vs^{s^r - r} = 7^{7^2 - 2} = 7^{47}, x = s^{s^r} = 7^{7^2} = 7^{49},$

$7^{49} = 2569235775210588780886114772242356213216 07,$

$7^{47} = 524333831675630363446145871886195145 5543$

$7^x = x^q \rightarrow 7^x = x^{7^{47}} \rightarrow 7^x = x^{524333831675630363446145871886195145 5543}$

$5. q = s^{vs^r - r} = 2^{2^r - r}\ , r\ \in \mathbb{N}, the\ solution\ is\ x = s^{vs^r} = 2^{2^r}.$