$$x^2 + y^2 = z^2 + w^2.$$

En parametriseret løsning af den Diofantiske ligning ovenfor.

Indledende bemærkning. *Udledningen baseres på*

(§): $x^2 + k^2 = y^2 \Leftrightarrow y^2 - x^2 = k^2$, *med* $x = p$ *og* $y = p + q$ *fås* $k^2 = 2pq + q^2$

(§§): $(n, m) \in \mathbb{Z}^2 \wedge nm \equiv 0\ Mod(i) \Longrightarrow \exists (t, u) \in \mathbb{N}^2 : i = tu \wedge t > 1 \wedge \big(n \equiv 0\ Mod(t) \vee m \equiv 0\ Mod(t)\big).$

$x^2 + y^2 = z^2 + w^2 \Leftrightarrow x^2 + y^2 - z^2 = w^2$

$x^2 + y^2 - z^2$ *er kvadrattal* $\Leftrightarrow \exists i \in \mathbb{Z}: y^2 - z^2 = 2xi + i^2 \Leftrightarrow (y+z)(y-z) = 2xi + i^2$, *jf.* (§).

$(y+z)(y-z) = 2xi + i^2 \Longrightarrow \exists (t, u, r) \in \mathbb{Z}^3 : i = tu \wedge t > 1 \wedge (y + z = tr \vee y - z = tr)$, *jf* (§§).

$i = tu$ (&). *Det ses at* $i \equiv 1\ Mod(2) \Longrightarrow t \equiv 1\ Mod(2) \wedge u \equiv 1\ Mod(2)$. (&&)

Hvis i *er et primtal sættes* $u = 1$ *og* $t = i$. *Da er* $2xi = 2tu$ *og*

$(y+z)(y-z) = 2xi + i^2 \Leftrightarrow y^2 - z^2 - t^2u^2 = 2tux$. *Der opdeles i to tilfælde.*

$\boldsymbol{A: y + z = tr \Longrightarrow z = tr - y.} \qquad \boldsymbol{B: y - z = tr \Longrightarrow z = tr + y.}$

A: Da er

$(y+z)(y-z) = tr(y-z) = 2xi + i^2$ (&&&)

$2tux = y^2 - z^2 - t^2u^2 = y^2 - (tr - y)^2 - t^2u^2 = -t^2r^2 + 2try - t^2u^2 = t(-tr^2 + 2ry - tu^2)$

$2tux = t(-tr^2 + 2ry - tu^2) \Leftrightarrow x = \frac{-tr^2 + 2ry - tu^2}{2u}$, *da er* $x = \frac{-tu}{2} + \frac{r(2y - tr)}{2u}$

Der søges et parametriseret $y \in \mathbb{Z}$ *så* $x \in \mathbb{Z}$. *Opdeling i tilfælde*

$A1: i \equiv 1\ Mod(2)$ *og* $A2: i \equiv 0\ Mod(2)$.

$\boldsymbol{A1: i \equiv 1\ Mod(2).}$

$i \equiv 1\ Mod(2) \Longrightarrow t \equiv 1\ Mod(2) \wedge u \equiv 1\ Mod(2) \wedge r \equiv 1\ Mod(2)$, *jf.* (&&) *og* (&&&)

Lad $y = \frac{tr + su}{2}$ *med* $s \equiv 1\ Mod(2)$, *da er* $x = \frac{-tu}{2} + \frac{r(2y - tr)}{2u} = \frac{-tu}{2} + \frac{r(tr + su - tr)}{2u} = \frac{sr - tu}{2}$

$$z = tr - y = tr - \frac{tr + su}{2} = \frac{tr - su}{2}$$

$$w^2 = x^2 + y^2 - z^2 = \left(\frac{sr - tu}{2}\right)^2 + \left(\frac{tr + su}{2}\right)^2 - \left(\frac{tr - su}{2}\right)^2 = \frac{1}{4}(4trsu + s^2r^2 + t^2u^2 - 2trsu)$$

$$w^2 = \frac{1}{4}(s^2r^2 + t^2u^2 + 2trsu) = \frac{1}{4}(sr + tu)^2 \Longrightarrow w = \frac{sr + tu}{2}$$

Tilfælde A1 resulterer derfor i nedenstående løsning.

$$x = \frac{sr - tu}{2}\ , y = \frac{tr + su}{2}, z = \frac{tr - su}{2}\ , w = \frac{sr + tu}{2}$$

$x^2 + y^2 = z^2 + w^2$. *Parametriserede løsninger.*

Heine Strømdahl, 24 − 10 − 2025.

Verificeret i GeoGebra↓↓↓

1	x=(s r-t u)/2 → $x = \frac{1}{2}\,(r\,s - t\,u)$
2	y=(t r+s u)/2 → $y = \frac{1}{2}\,(r\,t + s\,u)$
3	z=(t r-s u)/2 → $z = \frac{1}{2}\,(r\,t - s\,u)$
4	w=(s r+t u)/2 → $w = \frac{1}{2}\,(r\,s + t\,u)$
5	x^2+y^2-z^2-w^2 → $-w^2 + x^2 + y^2 - z^2$
6	(1 / 2 (r s - t u))^2+(1 / 2 (r t + s u))^2 - (1 / 2 (r t - s u))^2-(1 / 2 (r s + t u))^2 → 0