## An Even Simpler Proof of the Right-Hand Rule

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Suppose you are stranded at great altitude on the positive $z$-axis of a standard ("right-handed") Cartesian coordinate system (see Figure 1), looking vertiginously down on the vast, flat $xy$-plane (imagine being the first person to see Kansas from Mt. Everest). As hypothermia and hypoxia set in, you ask yourself, "How did I end up here? And is there a proof of the right-hand rule that requires nothing more difficult than high school trigonometry?"

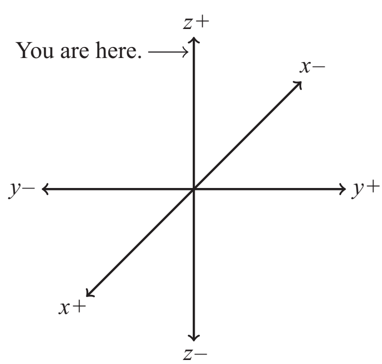


**Figure 1.** Stranded.

Your first question is unanswerable, but before we demonstrate that the answer to your second question is "yes," recall that the cross product of two vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ is denoted $\mathbf{a} \times \mathbf{b}$ and defined analytically as the vector $\langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle$. It is easy to prove directly from this definition that $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$ and, if either $\mathbf{a}$ or $\mathbf{b}$ is the zero vector or $\mathbf{a}$ is a multiple of $\mathbf{b}$, then $\mathbf{a} \times \mathbf{b}$ is the zero vector. Otherwise, there are two vectors of the correct length but of opposite directions that are orthogonal to $\mathbf{a}$ and $\mathbf{b}$. The right-hand rule states that to determine the direction of $\mathbf{a} \times \mathbf{b}$, place your right hand

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so that the little-finger edge of your hand is parallel to **a**, curl your fingers toward **b**, and move your thumb away from your other fingers. Then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

Several proofs of the right-hand rule have been published, e.g., [**1**, p. 192], [**2**], [**3**, ch. 7], [**4**, p. 616], of which Gao [**2**] is the simplest. Here is an even simpler proof.

Let $\mathbf{a_p} = \langle a_1, a_2, 0\rangle$ and $\mathbf{b_p} = \langle b_1, b_2, 0\rangle$ be the projections of **a** and **b** onto the $xy$-plane, respectively. First, assume that $\mathbf{a_p}$ is not a multiple of $\mathbf{b_p}$, i.e., that the plane that contains **a** and **b** is not perpendicular to the $xy$-plane. From the perspective of the positive $z$-axis, we will determine if **a** is to the right or left of **b** by the following procedure: First, look at the smaller of the two angles that the vectors form with each other. Then, without crossing your arms, move your arms so that one arm is parallel to **a** and move your other arm so that it is parallel to **b**. If your right arm is parallel to **a**, then **a** is to the right of **b** and, if your left arm is parallel to **a**, then **a** is to the left of **b**.

Now, the right-hand rule says that from the perspective of the positive $z$-axis, if **a** is to the right of **b**, then the $z$-component of $\mathbf{a} \times \mathbf{b}$ will be positive since your thumb will point up and, if **a** is to the left of **b**, then the $z$-component of $\mathbf{a} \times \mathbf{b}$ will be negative since your thumb will point down. Note that the left-right orientation of $\mathbf{a_p}$ and $\mathbf{b_p}$ is the same as the left-right orientation of **a** and **b**, i.e., if **a** is to the right of **b**, then $\mathbf{a_p}$ is to the right of $\mathbf{b_p}$, and if **a** is to the left of **b**, then $\mathbf{a_p}$ is to the left of $\mathbf{b_p}$.

Let $\alpha$ and $\beta$ be the measures of the angles that $\mathbf{a_p}$ and $\mathbf{b_p}$ form with the positive $x$-axis, respectively, where $0 \leq \alpha, \beta < 2\pi$ and, as usual, positive angle measurements correspond to counterclockwise rotations. Note that $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a_p} \times \mathbf{b_p}$ have the same $z$-component, namely

$$\begin{aligned} a_1 b_2 - a_2 b_1 &= \left(\left|\mathbf{a_p}\right| \cos\alpha\right)\left(\left|\mathbf{b_p}\right| \sin\beta\right) - \left(\left|\mathbf{a_p}\right| \sin\alpha\right)\left(\left|\mathbf{b_p}\right| \cos\beta\right) \\ &= \left|\mathbf{a_p}\right| \left|\mathbf{b_p}\right| (\sin\beta\, \cos\alpha - \sin\alpha\, \cos\beta) \\ &= \left|\mathbf{a_p}\right| \left|\mathbf{b_p}\right| \sin(\beta - \alpha). \end{aligned}$$

Suppose **a** is to the right of **b**. Then $\mathbf{a_p}$ is to the right of $\mathbf{b_p}$, so $0 < \beta - \alpha < \pi$ (see Figure 2(a)) or $-2\pi < \beta - \alpha < -\pi$ (see Figure 2(b)). In both cases, $\sin(\beta - \alpha) > 0$, so the $z$-component of $\mathbf{a} \times \mathbf{b}$ is positive, as stated by the right-hand rule. Similarly, if **a** is to the left of **b** then $\mathbf{a_p}$ is to the left of $\mathbf{b_p}$, so $-\pi < \beta - \alpha < 0$ or $\pi < \beta - \alpha < 2\pi$,

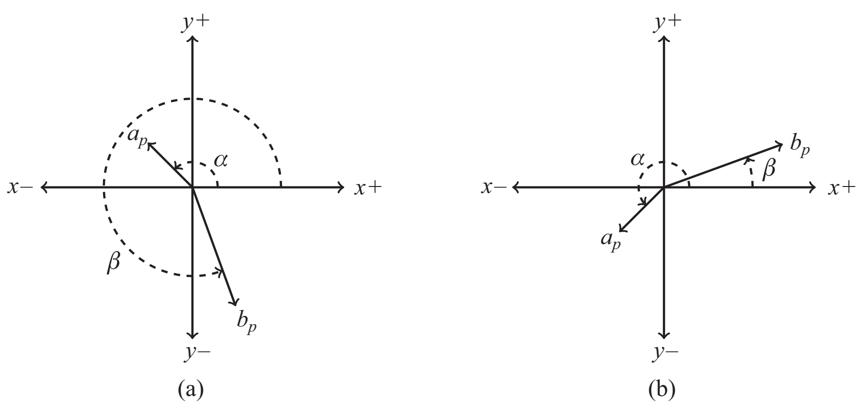


**Figure 2.** (a) The view from the positive $z$-axis, where $\mathbf{a_p}$ is to the right of $\mathbf{b_p}$ and $0 < \beta - \alpha < \pi$. (b) As in (a) with $-2\pi < \beta - \alpha < -\pi$.

 

so $\sin(\beta - \alpha) < 0$, thus the $z$-component is negative. This concludes the proof of the right-hand rule when $\mathbf{a_p}$ is not a multiple of $\mathbf{b_p}$.

If $\mathbf{a_p}$ is a multiple of $\mathbf{b_p}$, the proof is almost identical, except that we redefine $\mathbf{a_p}$ and $\mathbf{b_p}$ by letting $\mathbf{a_p} = \langle 0, a_2, a_3 \rangle$ and $\mathbf{b_p} = \langle 0, b_2, b_3 \rangle$ be the projections of $\mathbf{a}$ and $\mathbf{b}$ onto the $yz$-plane, respectively. Let $\alpha$ and $\beta$ be the angles that $\mathbf{a_p}$ and $\mathbf{b_p}$ make with the positive $y$-axis, respectively, where again $0 \leq \alpha, \beta < 2\pi$, but now positive angle measurements correspond to counterclockwise rotations as seen from a vantage point on the positive $x$-axis. Then the right-hand rules says that, from the perspective of the positive $x$-axis, the $x$-component of $\mathbf{a} \times \mathbf{b}$ is positive if and only if $\mathbf{a}$ is to the right of $\mathbf{b}$. Again, the left-right orientation of $\mathbf{a_p}$ and $\mathbf{b_p}$ is the same as the left-right orientation of $\mathbf{a}$ and $\mathbf{b}$. Now, $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a_p} \times \mathbf{b_p}$ have the same $x$-component, and the proof proceeds as above.

**Summary.** We present a trigonometric proof of the right-hand rule.

## References

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