VEKTORI

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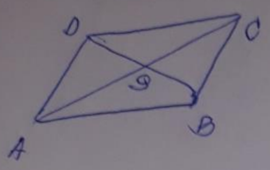
1. U paralelogramu ABCD odredi središte S i izračunaj: $\vec{BC}-\vec{DC}$, $\vec{AB}-\vec{BC}$, $\vec{AS}-\vec{BS}$, $\vec{BS}-\vec{SD}$, $\vec{AC}-\vec{SC}$, $\vec{AS}-\vec{SD}$



(1) $\vec{BC}-\vec{DC}=$
$=\vec{BC}+\vec{CD}=\vec{BD}$

(2) $\vec{AB}-\vec{BC}=$
$=\vec{AB}+\vec{CB}=$
$=\vec{AB}+\vec{DA}=$
$=\vec{DA}+\vec{AB}=\vec{DB}$

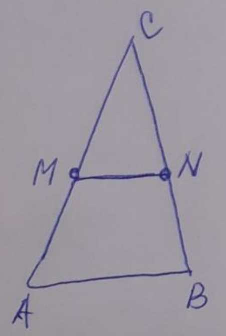
(3) $\vec{AS}-\vec{BS}=\vec{AS}+\vec{SB}$
$=\vec{AB}$

(4) $\vec{BS}-\vec{SD}=$
$=\vec{BS}+\vec{DS}=$
$=\vec{SD}+\vec{DS}=\vec{0}$

(5) $\vec{AC}-\vec{SC}=$
$=\vec{AC}+\vec{CS}=$
$=\vec{AS}+\vec{SC}+\vec{CS}=\vec{AS}$

(6) $\vec{AS}-\vec{SD}=$
$=\vec{AS}+\vec{DS}=$
$=\vec{AS}+\vec{SB}=\vec{AB}$

2. Srednjica trokuta je dužina koja spaja polovišta dviju stranica, paralelna je sa trećom stranicom i upola kraća od nje. DOKAŽI!



$\overline{MN}$ - srednjica $\triangle ABC$, M i N polovišta od $\overline{AC}$ i $\overline{BC}$

$\overline{MN} \parallel \overline{AB}$ i $|\overline{MN}|=\frac{1}{2}|\overline{AB}|$ — DOKAŽI!

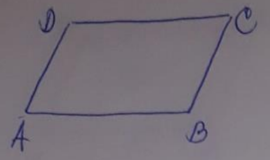
$$\vec{AM}+\vec{MN}+\vec{NB}=\vec{AB}$$
$$\vec{MN}=\vec{MC}+\vec{CN} \quad /+$$
$$\vec{AM}+\vec{MN}+\vec{NB}+\vec{MN}=\vec{AB}+\vec{MC}+\vec{CN}$$
$$\vec{AM}+2\vec{MN}+\vec{NB}=\vec{AB}+\vec{MC}+\vec{CN}$$

- kako je $\vec{AM}=\vec{MC}$; $\vec{CN}=\vec{NB}$

$$\Rightarrow 2\vec{MN}=\vec{AB}$$
$$\vec{MN}=\frac{1}{2}\vec{AB}$$

($\vec{AB}$ i $\vec{MN}$ - kolinearni)

3. U paralelogramu ABCD prikaži vektor $\overrightarrow{AD}$ kao linearnu kombinaciju vektora $\overrightarrow{AC}$ i $\overrightarrow{BD}$



$$\overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC}$$
$$\overrightarrow{AD} + \overrightarrow{AB} = \overrightarrow{AC}$$
$$\overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{AB}$$
$$\overrightarrow{AD} + \overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{AC}$$
$$2\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{BD}$$
$$\overrightarrow{AD} = \frac{1}{2}(\overrightarrow{AC} + \overrightarrow{BD})$$

4. Vektori $\vec{m}$ i $\vec{n}$ su nekolinearni. Odredi realni broj X tako da vektori $\vec{a} = (x-1)\vec{m} + \vec{n}$ i $\vec{b} = 3\vec{m} + (x+1)\vec{n}$ budu kolinearni.

$$\vec{a} = (x-1)\vec{m} + \vec{n} \qquad \vec{a} = k \cdot \vec{b}$$
$$\vec{b} = 3\vec{m} + (x+1)\vec{n}$$

$$(x-1)\vec{m} + \vec{n} = k \cdot (3\vec{m} + (x+1)\vec{n}) \rightarrow (x-1)\vec{m} + \vec{n} = 3k\vec{m} + (x+1)k\vec{n}$$

$\Rightarrow$ $x - 1 = 3k$ i $1 = (x+1)k$

$$x = 3k + 1 \longrightarrow 1 = (3k+1+1) \cdot k$$
$$1 = 3k^2 + 2k$$
$$3k^2 + 2k - 1 = 0$$
$$k_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 3 \cdot 1}}{2 \cdot 3}$$
$$k_{1,2} = \frac{-2 \pm 4}{6} \Rightarrow k_1 = -1 \text{ i } k_2 = \frac{1}{3}$$
$$\text{a } x_1 = -2, \quad x_2 = 2$$

za $\boxed{x = -2}$ $\Rightarrow$ $\vec{a} = (-2-1)\vec{m} + \vec{m}$ i $\vec{b} = 3\vec{m} + (-2+1)\vec{n}$

$\vec{a} = -3\vec{m} + \vec{n}$ $\qquad$ $\vec{b} = 3\vec{m} - \vec{n}$

$$\vec{b} = -\vec{a}$$

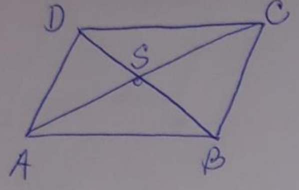
za $\boxed{x = 2}$ $\quad$ $\vec{a} = (2-1)\vec{m} + \vec{n}$ i $\vec{b} = 3\vec{m} + (2+1)\vec{n}$

$\vec{a} = \vec{m} + \vec{n}$ $\qquad$ $\vec{b} = 3\vec{m} + 3\vec{n}$

$$\vec{b} = 3(\vec{m} + \vec{n})$$

$$\vec{b} = 3 \cdot \vec{a}$$

5. Točke $A(0,3)$ i $B(2,2)$ su dva vrha paralelograma a točka $S(3,4)$ je sjecište njegovih dijagonala. Odredi koordinate ostala dva vrha C i D.



$A(0,3)$, $B(2,2)$, $S(3,4)$

- vrijedi $|\overline{AS}| = |\overline{SC}|$, odnosno $\vec{AS} = \vec{SC}$
i $|\overline{BS}| = |\overline{SD}|$, odnosno $\vec{BS} = \vec{SD}$

(1)
$$\vec{AS} = (x_S - x_A)\vec{i} + (y_S - y_A)\vec{j}$$
$$\vec{SC} = (x_C - x_S)\vec{i} + (y_C - y_S)\vec{j}$$

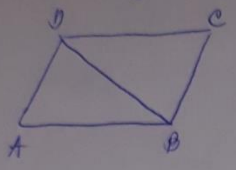
$$(x_S - x_A)\vec{i} + (y_S - y_A)\vec{j} = (x_C - x_S)\vec{i} + (y_C - y_S)\vec{j}$$
$$(3-0)\vec{i} + (4-3)\vec{j} = (x_C - 3)\vec{i} + (y_C - 4)\vec{j}$$
$$3\vec{i} + \vec{j} = (x_C - 3)\vec{i} + (y_C - 4)\vec{j}$$

$x_C - 3 = 3$ i $y_C - 4 = 1$ $\Rightarrow$ $C(6,5)$
$x_C = 6$ $y_C = 5$

(2) $\vec{BS} = \vec{SD}$
$$(x_S - x_B)\vec{i} + (y_S - y_B)\vec{j} = (x_D - x_S)\vec{i} + (y_D - y_S)\vec{j}$$
$$(3-2)\vec{i} + (4-2)\vec{j} = (x_D - 3)\vec{i} + (y_D - 4)\vec{j}$$
$$\vec{i} + 2\vec{j} = (x_D - 3)\vec{i} + (y_D - 4)\vec{j}$$

$x_D - 3 = 1$ i $y_D - 4 = 2$ $\Rightarrow$ $D(4,6)$
$x_D = 4$ $y_D = 6$

6. Točke $A(-1,-1)$, $B(3,-2)$ i $C(5,2)$ su tri uzastopna vrha paralelograma ABCD. Kolika je duljina dijagonale $\overline{BD}$?



$$\vec{BD} = \vec{BC} + \vec{CD}$$

$$D(x_D, y_D) = ?$$

- iz $\vec{AD} = \vec{BC}$ slijedi:

$$(x_D - x_A)\vec{i} + (y_D - y_A)\vec{j} = (x_C - x_B)\vec{i} + (y_C - y_B)\vec{j}$$

$$(x_D + 1)\vec{i} + (y_D + 1)\vec{j} = (5-3)\vec{i} + (2+2)\vec{j}$$

$$(x_D + 1)\vec{i} + (y_D + 1)\vec{j} = 2\vec{i} + 4\vec{j}$$

$x_D + 1 = 2$ i $y_D + 1 = 4$

$x_D = 1$ $\quad y_D = 3$ $\Rightarrow$ $D(1,3)$

$$|\vec{BD}| = \sqrt{(x_D - x_B)^2 + (y_D - y_B)^2} = \sqrt{(1-3)^2 + (3+2)^2} =$$

$$= \sqrt{(-2)^2 + 5^2} = \sqrt{4+25} = \sqrt{29}$$

7. Odredi apscisu točke $O(x_0, 1)$ tako da točka O pripada pravcu $p(P,R)$; $P(-4,-3)$, $R(2,0)$

$O(x_0, 1)$ — vrijedi kolinearnost $\vec{PO} = k \cdot \vec{OR}$

$P(-4,-3)$

$R(2,0)$

$$\vec{PO} = k \cdot \vec{OR}$$

$$(x_0 - x_P)\vec{i} + (y_0 - y_P)\vec{j} = k \cdot \left((x_R - x_0)\vec{i} + (y_R - y_0)\vec{j}\right)$$

$$(x_0 + 4)\vec{i} + (1+3)\vec{j} = k\left((2 - x_0)\vec{i} + (0-1)\vec{j}\right)$$

$$(x_0 + 4)\vec{i} + 4\vec{j} = k(2 - x_0)\vec{i} - k\vec{j}$$

$-k = 4$ i $x_0 + 4 = k \cdot (2 - x_0)$

$k = -4$ $\quad x_0 + 4 = -4(2 - x_0)$

$$x_0 + 4 = -8 + 4x_0$$

$$3x_0 = 12$$

$$\underline{x_0 = 4}$$

$O(4,1)$

8. Koliko iznosi ordinata točke $A(2, y_A)$ ako je duljina vektora $\overrightarrow{AB}$ jednaka 3 i $B(-1, 2)$?

$A(2, y_A)$
$B(-1, 2)$
$|\overrightarrow{AB}| = 3$

$$|\overrightarrow{AB}| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$
$$3 = \sqrt{(-1-2)^2 + (2-y_A)^2} \quad /^2$$
$$9 = (-3)^2 + (2-y_A)^2$$
$$9 = 9 + 4 - 4y_A + y_A^2$$
$$y_A^2 - 4y_A + 4 = 0$$
$$y_{A_{1,2}} = \frac{4 \pm \sqrt{16-16}}{2}$$
$$\underline{y_A = 2}$$

$A(2, 2)$

9. Ako je $\vec{a} = 2\vec{i} - \vec{j}$ i $\vec{b} = 3\vec{i} + 2\vec{j}$, odredi $\vec{c}$ tako da vrijedi: $\vec{a} \cdot \vec{c} = 7$ i $\vec{b} \cdot \vec{c} = 7$

$\vec{a} = 2\vec{i} - \vec{j}$
$\vec{b} = 3\vec{i} + 2\vec{j}$
$\vec{a} \cdot \vec{c} = 7$
$\vec{b} \cdot \vec{c} = 7$

(1) $\vec{a} \cdot \vec{c} = 7$
$$(2\vec{i} - \vec{j})(x_c\vec{i} + y_c\vec{j}) = 7$$
$$2x_c\vec{i}^2 + 2y_c\vec{i}\vec{j} - x_c\vec{j}\vec{i} - y_c\vec{j}^2 = 7$$
$$\underline{2x_c - y_c = 7}$$

$\vec{i}^2 = 1, \vec{j}^2 = 1$
$\vec{i} \cdot \vec{j} = 0$

(2) $\vec{b} \cdot \vec{c} = 7$
$$(3\vec{i} + 2\vec{j}) \cdot (x_c\vec{i} + y_c\vec{j}) = 7$$
$$3x_c\vec{i}^2 + 3y_c\vec{i}\vec{j} + 2x_c\vec{j}\vec{i} + 2y_c\vec{j}^2 = 7$$
$$\underline{3x_c + 2y_c = 7}$$

(1) i (2)
$$2x_c - y_c = 7$$
$$3x_c + 2y_c = 7$$
$$2x_c - y_c = 3x_c + 2y_c$$
$$-x_c = 3y_c$$
$$\underline{x_c = -3y_c}$$
$$2 \cdot (-3y_c) - y_c = 7$$
$$-6y_c - y_c = 7$$
$$-7y_c = 7$$
$$\boxed{y_c = -1}$$

$$x_c = -3 \cdot (-1)$$
$$\boxed{x_c = 3}$$

$$\vec{c} = x_c\vec{i} + y_c\vec{j}$$
$$\boxed{\vec{c} = 3\vec{i} - \vec{j}}$$

10. Zadani su vektori: $\vec{a} = 3\vec{i} + 4\vec{j}$ i $\vec{b} = -5\vec{i} + 2\vec{j}$.
Koliki kut zatvaraju ti vektori?

$$\vec{a} = 3\vec{i} + 4\vec{j}$$
$$\vec{b} = -5\vec{i} + 2\vec{j}$$
$$\varphi = ?$$

$$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$$

$$\cos\varphi = \frac{(3\vec{i} + 4\vec{j})(-5\vec{i} + 2\vec{j})}{\sqrt{x_a^2 + y_a^2} \cdot \sqrt{x_b^2 - y_b^2}}$$

$$\cos\varphi = \frac{-15\vec{i}^2 + 6\vec{i}\vec{j} - 20\vec{i}\vec{j} + 8\vec{j}^2}{\sqrt{3^2 + 4^2} \cdot \sqrt{(-5)^2 + 2^2}}$$

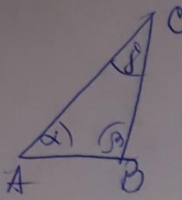
$$\cos\varphi = \frac{-15 + 8}{\sqrt{9 + 16} \cdot \sqrt{25 + 4}}$$

$$\cos\varphi = \frac{-7}{5 \cdot \sqrt{29}}$$

$$\varphi = 105^\circ 04' 07''$$

11. Odredi najveći kut u ΔABC čiji su vrhovi:

$A(-1,3),\ B(1,1)\ C(5,3)$



- najveći kut u trokutu nalazi se nasuprot najdulje stranice
- tražimo $\overline{AB}$, $\overline{AC}$ i $\overline{BC}$

$$\vec{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} = (1+1)\vec{i} + (1-3)\vec{j} = 2\vec{i} - 2\vec{j}$$

$$|\vec{AB}| = \sqrt{x_A^2 + y_A^2} = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$\vec{AC} = (x_C - x_A)\vec{i} + (y_C - y_A)\vec{j} = (5+1)\vec{i} + (3-3)\vec{j} = 6\vec{i}$$

$$|\vec{AC}| = \sqrt{6^2} = 6$$

$$\vec{BC} = (x_C - x_B)\vec{i} + (y_C - y_B)\vec{j} = (5-1)\vec{i} + (3-1)\vec{j} = 4\vec{i} + 2\vec{j}$$

$$|\vec{BC}| = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

$\overline{AC}$ – najdulja stranica, znači najveći kut je β

$$\vec{BA} = -2\vec{i} + 2\vec{j}$$

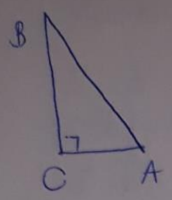
$$\cos\beta = \frac{\vec{BA} \cdot \vec{BC}}{|\vec{AB}| \cdot |\vec{BC}|}$$

$$\cos\beta = \frac{(-2i+2j)\cdot(4i+2\vec{j})}{2\sqrt{2} \cdot 2\sqrt{5}} = \frac{-8\vec{i}^2 - 4\vec{i}\vec{j} + 8\vec{j}\vec{i} + 4\vec{j}^2}{4\sqrt{10}} = \frac{-8+4}{4\sqrt{10}} = \frac{-4}{4\sqrt{10}}$$

$$\cos\beta = -\frac{1}{\sqrt{10}}$$

$$\underline{\beta = 108^\circ 26' 06''}$$

12. Vrh C pravokutnog trokuta nalazi se na osi apscisa, a ostala dva vrha hipotenuze su $A(4,-3)$ i $B(1,6)$. Odredi vrh C.



$$C(x_0,0)$$

$$\measuredangle C = 90^\circ,$$

$$\cos\measuredangle C = \frac{\vec{CA}\cdot\vec{CB}}{|\vec{CA}|\cdot|\vec{CB}|}$$

$$0 = \frac{\vec{CA}\cdot\vec{CB}}{|\vec{CA}|\cdot|\vec{CB}|} \Rightarrow \vec{CA}\cdot\vec{CB}=0$$

$$\vec{CA}\cdot\vec{CB}=0$$

$$\left((x_A-x_c)\vec{i}+(y_A-y_c)\vec{j}\right)\cdot\left((x_B-x_c)\vec{i}+(y_B-y_c)\vec{j}\right)=0$$

$$\left((4-x_c)\vec{i}+(-3-0)\vec{j}\right)\cdot\left((1-x_c)\vec{i}+(6-0)\vec{j}\right)=0$$

$$(4-x_c)(1-x_c)\vec{i}^2+(4-x_c)\cdot(6)\vec{j}\vec{i}-3\vec{j}(1-x_c)\vec{i}-3\cdot 6\vec{j}^2=0$$

$$4-4x_c-x_c+x_c^2+(24-6x_c)\cdot 0-(3-3x_c)\vec{j}\vec{i}-18\vec{j}^2=0$$

$$x^2-5x_c+4-18=0$$

$$x_c^2-5x_c-14=0$$

$$x_{c_{1,2}}=\frac{5\pm\sqrt{25+56}}{2}$$

$$x_{c_{1,2}}=\frac{5\pm 9}{2}$$

$$x_{c_1}=-2,\quad x_{c_2}=7$$

$$C_1(-2,0),\quad C_2(7,0)$$

13. Ako je $|\vec{a}|=5$, $|\vec{a}+\vec{b}|=13$, $|\vec{a}-\vec{b}|=9$

Kolika je duljina vektora $\vec{b}$?

$|\vec{a}|=5$

$|\vec{a}+\vec{b}|=13$

$|\vec{a}-\vec{b}|=9$

$|\vec{b}|=?$

$|\vec{a}+\vec{b}|=\sqrt{(\vec{a}+\vec{b})^2}\ \Big/^2$; $|\vec{a}-\vec{b}|=\sqrt{(\vec{a}-\vec{b})^2}\ \Big/^2$

$|\vec{a}+\vec{b}|^2=(\vec{a}+\vec{b})^2$

$13^2=\vec{a}^2+2\vec{a}\vec{b}+\vec{b}^2$

$|\vec{a}-\vec{b}|^2=(\vec{a}-\vec{b})^2$

$9^2=\vec{a}^2-2\vec{a}\vec{b}+\vec{b}^2$

$\vec{a}^2+2\vec{a}\vec{b}+\vec{b}^2=169$

$\vec{a}^2-2\vec{a}\vec{b}+\vec{b}^2=81 \quad \Big/ +$

$2\vec{a}^2+2\vec{b}^2=250$

$2\cdot 5^2+2\vec{b}^2=250$

$2\cdot 25+2\vec{b}^2=250$

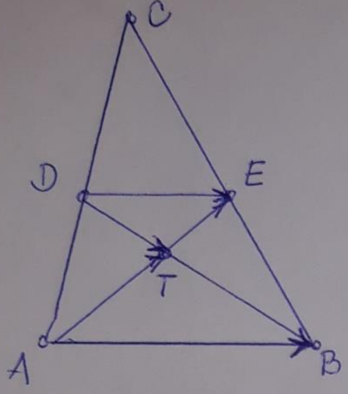
$50+2\vec{b}^2=250$

$2\vec{b}^2=200 \quad /:2$

$\vec{b}^2=100 \quad /\sqrt{\ }$

$|\vec{b}|=10$

(*) Dokaži da težište (T) trokuta (ABC) dijeli težišnicu u omjeru 2:1, računajući od vrha trokuta.



T - težište $\triangle ABC$

$\overline{DE}$ - srednjica $\triangle ABC$, $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AB}$

$\overrightarrow{AT} + \overrightarrow{TB} = \overrightarrow{AB}$

$\overrightarrow{DT} + \overrightarrow{TE} = \overrightarrow{DE}$

$\overrightarrow{AB} = \overrightarrow{AT} + \overrightarrow{TB} \Rightarrow \overrightarrow{DE} = \frac{1}{2}\left(\overrightarrow{AT} + \overrightarrow{TB}\right)$

$\downarrow$

$\overrightarrow{DT} + \overrightarrow{TE} = \frac{1}{2}\overrightarrow{AT} + \frac{1}{2}\overrightarrow{TB}$

$\overrightarrow{DT} - \frac{1}{2}\overrightarrow{TB} = \overrightarrow{ET} - \frac{1}{2}\overrightarrow{TA}$

- vektori s lijeve strane jednakosti nisu paralelni, zato je ta jednakost moguća samo ako je:

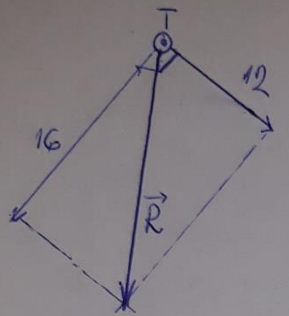
$\overrightarrow{DT} - \frac{1}{2}\overrightarrow{TB} = \vec{0}$ i $\overrightarrow{ET} - \frac{1}{2}\overrightarrow{TA} = \vec{0}$

$\overrightarrow{DT} = \frac{1}{2}\overrightarrow{TB}$ $\qquad$ $\overrightarrow{ET} = \frac{1}{2}\overrightarrow{TA}$

$\overrightarrow{TB} = 2\overrightarrow{DT}$ $\qquad$ $\overrightarrow{TA} = 2\overrightarrow{ET}$

$|\overrightarrow{TB}| = 2 \cdot |\overrightarrow{DT}|$ $\qquad$ $|\overrightarrow{TA}| = 2|\overrightarrow{ET}|$

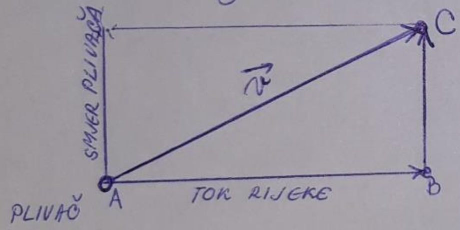
(*) Dvije sile od 12 N i 16 N djeluju na neko tijelo pod pravim kutom. Odredi intenzitet rezultante (smjer i veličinu).



$$|\vec{R}| = \sqrt{12^2+16^2} = \sqrt{144+256} = \sqrt{400} = 20$$

$$|\vec{R}| = 20\ N$$

(*) Plivač pliva brzinom 5 km/h okomito na tok rijeke. Rijeka teče brzinom od 12 km/h i mijenja brzinu plivača i smjer plivanja. Odredi intenzitet i smjer rezultirajuće brzine $\vec{v}$.

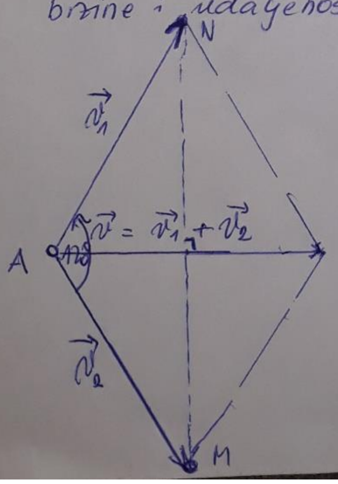


$$\vec{AB} + \vec{BC} = \vec{v}$$

$$|\vec{v}| = \sqrt{12^2+5^2} = \sqrt{144+25} = \sqrt{169} = 13$$

$$|\vec{v}| = 13$$

(*) U istom trenutku dva zrakoplova polaze iz istog mjesta A i lete pravocrtno brzinom $v_1 = v_2 = 400$ km/h. Pravci koji označavaju put leta aviona čine ∡ 120°. Odredi smjer, intenzitet rezultata brzine i udaljenost zrakoplova nakon 2 sata leta.



$\vec{v} = \vec{v}_1 + \vec{v}_2$, $|\vec{v}| = |\vec{v}_1| = |\vec{v}_2| = 400$ km/h

$d = \overline{MN}$ – udaljenost nakon dva sata

$$d = 2 \cdot \frac{2v\sqrt{3}}{2}$$

$$d = 2v\sqrt{3} = 2 \cdot \cancel{h} \cdot 400\,km/\cancel{h}\,\sqrt{3}$$

$$d = 800\sqrt{3}\ km$$