PROJECTILE MOTION *Parabolic trajectory*

From the kinematic equations we have

$$x(t) = v_0 \cos(\theta)\, t \qquad y(t) = y_0 + v_0 \sin(\theta)\, t - \frac{1}{2} g t^2$$

We seek the cartesian form of this motion, y(x). Solving the easier equation for t, $t = \frac{x}{v_0 \cos(\theta)}$

we then use this in y(t) to get

$$y(x) = y_0 + \frac{v_0 \sin(\theta)\, x}{v_0 \cos(\theta)} - \frac{1}{2} g \left(\frac{x}{v_0 \cos(\theta)}\right)^2$$

$$y(x) := \left[\frac{-g}{2\left(v_0 \cos(\theta)\right)^2} x^2 + \tan(\theta)\, x + y_0\right] \quad (1)$$

This is a quadratic in x, with the usual notation defined as

$$a = \frac{-g}{2\left(v_0 \cos(\theta)\right)^2} \qquad b = \tan(\theta) \qquad c = y_0$$

It can be shown, using the analysis of the general second-order function

$$A x^2 + B x y + C y^2 + D x + E y + F = 0$$

that Eq(1) describes a parabola. However, it is useful to develop Eq(1) into another form, which will make this clearer. Completing the square in Eq(1), we have

$$y(x) = a x^2 + b x + c \qquad \frac{y}{a} = x^2 + \frac{b}{a} x + \frac{c}{a} \qquad \frac{y}{a} = x^2 + \frac{b}{a} x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}$$

$$\frac{y}{a} = \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2 \qquad y(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

This is now in the general form $y = a(x - h)^2 + k$ which is the "vertex" form of a parabola, with vertex at x = h and y = k.

This can be quickly checked using calculus; returning to the original form,

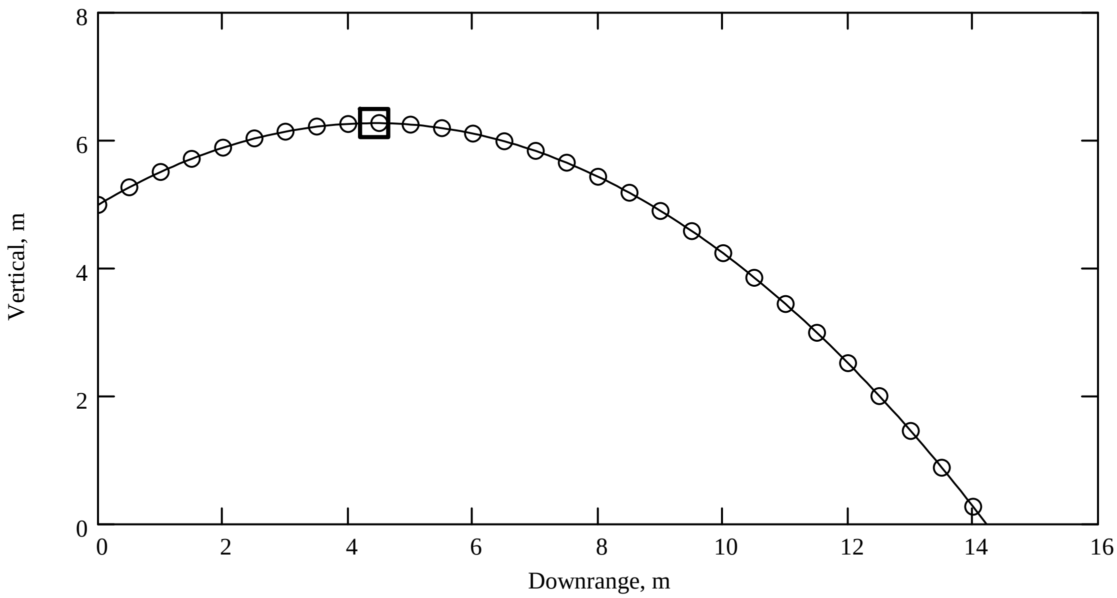
$$y(x) = a x^2 + b x + c \qquad \frac{dy}{dx} = 2 a x + b \qquad 0 = 2 a x_V + b \qquad x_V = \frac{-b}{2a}$$

$$y_V = y(x_V) = a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c = c - \frac{b^2}{4a}$$

Substituting the original parameters and cleaning up yields the vertex form of the trajectory

$$yV(x) := \frac{-g}{2\left(v_0 \cos(\theta)\right)^2}\left(x - \frac{v_0^2}{2g}\sin(2\theta)\right)^2 + \left(y_0 + \frac{v_0^2}{2g}\sin(\theta)^2\right) \quad (2)$$

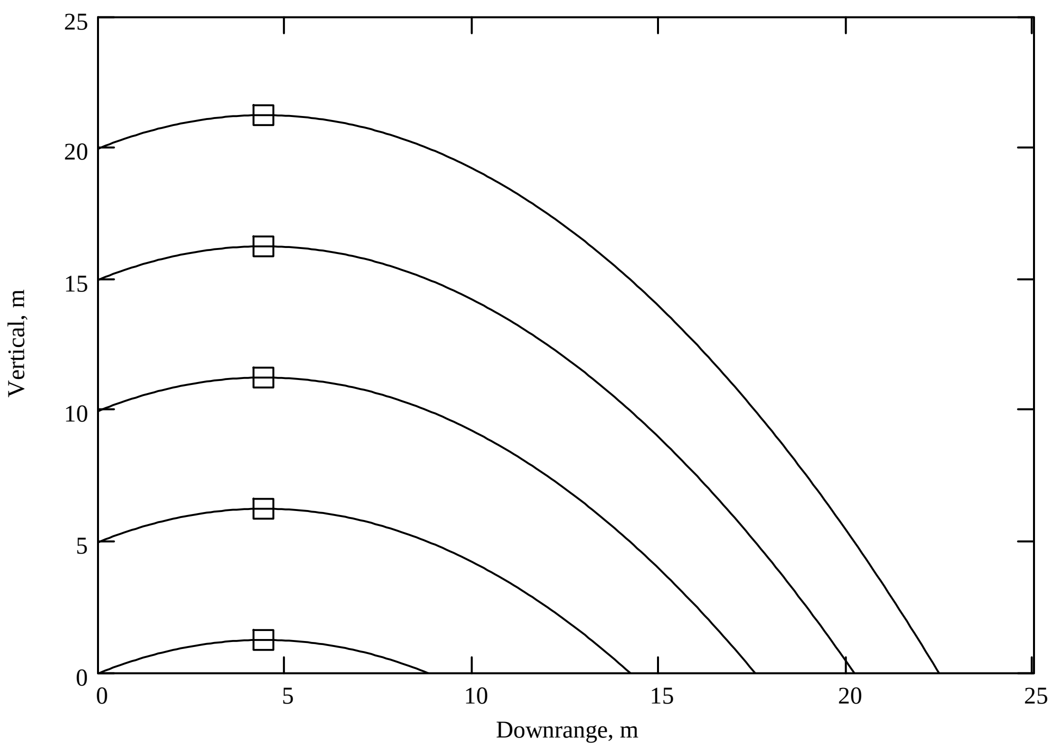
Let's plot these to verify that the trajectory is the same. The line is Eq(1), circles are Eq(2).



The position of the vertex of the trajectory (maximum height) is often of interest. We can read this directly from Eq(2), or use the definitions from the calculus. Either way we have

$$x_V := \frac{v_0^{\,2}}{2\,g}\sin(2\,\theta) \qquad y_V := y_0 + \frac{v_0^{\,2}}{2\,g}\sin(\theta)^2 \tag{3}$$

This position is shown on the plot as a square. Note that the x-position of the vertex does not depend on the initial y position. This is seen in the next plot.



We might also need to know the time it takes to reach the vertex position. An easy way to find this is to exploit the fact that the x-velocity is constant. Then, since we know the x-coordinate, the time will be

$$T_V = \frac{x_V}{v_0 \cos(\theta)} = \frac{v_0^{\,2}}{2\,g}\,\frac{\sin(2\,\theta)}{v_0 \cos(\theta)} = \frac{v_0}{g}\,\frac{\sin(\theta)\,\cos(\theta)}{\cos(\theta)} = \frac{v_0}{g}\,\sin(\theta)$$

Observe that (1) the x-coordinate of the vertex does not depend on the initial y; (2) the time required to reach the vertex also is independent of the initial y. As will be seen on another sheet, the time of flight when the initial height is zero is twice the time to reach the vertex.

Also note in Eq(1) or (2) that as the initial velocity increases, the quadratic term vanishes, and the function becomes linear. This is the "frozen rope" trajectory of, e.g., a baseball or football. In Eq(1) this is clear by inspection, in Eq(2) we should use a limit:

$$\lim_{v_0 \to \infty} \left[\frac{-g}{2\,(v_0 \cos(\theta))^2} \left(x - \frac{v_0^{\,2}}{2\,g} \sin(2\,\theta) \right)^2 + \left(y_0 + \frac{v_0^{\,2}}{2\,g} \sin(\theta)^2 \right) \right] \quad = \quad \tan(\theta)\,x + y_0$$

Finally, since we have a quadratic for the trajectory, the roots of that function would seem to be of interest. In fact, the positive root is called the "range" of the projectile, and this is discussed elsewhere.

In analytic geometry the parabola is defined as the locus of points equidistant from a point (called the "focus") and a line (called the "directrix"). The distance from the directrix, or the focus, to the vertex is called "p". For the orientation of a projectile trajectory (as opposed to a generic parabola), we will find the directrix and focus at

$$y_D = y_V + p \qquad\qquad y_F = y_V - p \qquad x_F = x_V = \frac{v_0^{\,2}}{2\,g} \sin(2\,\theta)$$

So the directrix is above the vertex and the focus is below it. The parameter p is obtained from the vertex form of the parabola (vertex at h,k):

$$4\,p\,y(x) = (x - h)^2 + k$$

This says $\quad \dfrac{1}{4\,p} = \dfrac{-g}{2\,(v_0 \cos(\theta))^2} \quad$ so that $\quad |p| = \dfrac{(v_0 \cos(\theta))^2}{2\,g}$

Then the equation of the directrix is

$$y_D = y_V + p = y_0 + \frac{v_0^{\,2}}{2\,g} \sin(\theta)^2 + \frac{(v_0 \cos(\theta))^2}{2\,g} = \frac{v_0^{\,2}}{2\,g} \left(\sin(\theta)^2 + \cos(\theta)^2 \right) + y_0$$

$$\boxed{y_D := y_0 + \frac{v_0^{\,2}}{2\,g}}$$

It is interesting to note that this is the vertical distance an object will travel if projected straight upward from an initial height y0. We can also see that the directrix is the same for any trajectory with a given initial velocity, regardless of the initial angle. The focus, on the other hand, is located at

$$x_F := \frac{v_0^{\,2}}{2\ g}\ \sin(2\ \theta) \qquad y_F := \frac{v_0^{\,2}}{2\ g}\left(1 - 2\ \cos(\theta)^2\right) + y_0$$

which clearly does depend on the angle. The plot below illustrates the directrix and focus.